UNIT-4: RANDOM PROCESSES: SPECTRAL CHARACTERISTICS

In this unit we will study the characteristics of random processes regarding correlation and covariance functions which are defined in time domain. This unit explores the important concept of characterizing random processes in the frequency domain. These characteristics are called spectral characteristics. All the concepts in this unit can be easily learnt from the theory of Fourier transforms.

Consider a random process X(t). The amplitude of the random process, when it varies randomly with time, does not satisfy Dirichlet’s conditions. Therefore it is not possible to apply the Fourier transform directly on the random process for a frequency domain analysis. Thus the autocorrelation function of a WSS random process is used to study spectral characteristics such as power density spectrum or power spectral density (psd).

**Power Density Spectrum:** The power spectrum of a WSS random process X(t) is defined as the Fourier transform of the autocorrelation function RXX(τ) of X(t). It can be expressed as

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega \tau} d\tau$$

We can obtain the autocorrelation function from the power spectral density by taking the inverse Fourier transform i.e.

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega \tau} d\omega$$

Therefore, the power density spectrum SXX(ω) and the autocorrelation function RXX(τ) are Fourier transform pairs.

The power spectral density can also be defined as

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

Where XT(ω) is a Fourier transform of X(t) in interval [-T,T]

**Average power of the random process:** The average power PXX of a WSS random process X(t) is defined as the time average of its second order moment or autocorrelation function at τ = 0.

Mathematically...
\[ P_{XX} = A \{E[X^2(t)]\} \]
\[ P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)] dt \]

Or \( P_{XX} = R_{XX}(\tau) \) | \( \tau = 0 \)

We know that from the power density spectrum
\[ R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega \tau} d\omega \]

At \( \tau = 0 \) \( P_{XX} = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \)

Therefore average power of \( X(t) \) is
\[ P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \]

**Properties of power density spectrum:** The properties of the power density spectrum \( S_{XX}(\omega) \) for a WSS random process \( X(t) \) are given as

1. \( S_{XX}(\omega) \geq 0 \)

   Proof: From the definition, the expected value of a non negative function

2. The power spectral density at zero frequency is equal to the area under the curve of the autocorrelation \( R_{XX}(\tau) \) i.e.
\[ S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau \]

   Proof: From the definition we know that
\[ S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega \tau} d\tau \] at \( \omega = 0 \).
\[ S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau) d\tau \]
3. The power density spectrum of a real process $X(t)$ is an even function i.e.

$$S_{XX}(-\omega) = S_{XX}(\omega)$$

Proof: Consider a WSS real process $X(t)$, then

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega \tau} d\tau$$

also

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{j\omega \tau} d\tau$$

Substitute $\tau = -\tau$ then

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(-\tau) e^{-j\omega \tau} d\tau$$

Since $X(t)$ is real, from the properties of autocorrelation we know that, $R_{XX}(-\tau) = R_{XX}(\tau)$

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{j\omega \tau} d\tau$$

4. $S_{XX}(\omega)$ is always a real function

5. If $S_{XX}(\omega)$ is a psd of the WSS random process $X(t)$, then

6. If $X(t)$ is a WSS random process with psd $S_{XX}(\omega)$, then the psd of the derivative of $X(t)$ is equal to $\omega^2$ times the psd $S_{XX}(\omega)$.

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$$

**Cross power density spectrum:** Consider two real random processes $X(t)$ and $Y(t)$, which are jointly WSS random processes, then the cross power density spectrum is defined as the Fourier transform of the cross correlation function of $X(t)$ and $Y(t)$ and is expressed as

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau$$

and

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j\omega \tau} d\tau$$

by inverse Fourier transformation, we can obtain the cross correlation functions as

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega \tau} d\omega$$

and

$$R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) e^{j\omega \tau} d\omega$$

The

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = A \{E[X^2(t)]\} = R_{XX}(0)$$

forms a Fourier transform pair.
The real part of \( S_{XY}(\omega) \) and real part \( S_{YX}(\omega) \) are even functions of \( \omega \). i.e.

\[ \text{Re} \left[ S_{XY}(\omega) \right] \text{ and } \text{Re} \left[ S_{YX}(\omega) \right] \text{ are even functions.} \]

**Properties of cross power density spectrum:** The properties of the cross power for real random processes \( X(t) \) and \( Y(t) \) are given by

1. \( S_{XY}(-\omega) = S_{XY}(\omega) \) and \( S_{YX}(-\omega) = S_{YX}(\omega) \)

**Proof:** Consider the cross correlation function \( R_{XY}(\tau) \). The cross power density spectrum is

\[ S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau \]

Let \( \tau = -\tau \) Then

\[ S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(-\tau) e^{j\omega \tau} d\tau \]

since \( R_{XY}(-\tau) = R_{XY}(\tau) \)

\[ S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau \]

Therefore \( S_{XY}(-\omega) = S_{XY}(\omega) \) Similarly \( S_{YX}(-\omega) = S_{YX}(\omega) \) hence proved.

2. The real part of \( S_{XY}(\omega) \) and real part \( S_{YX}(\omega) \) are even functions of \( \omega \) i.e. \( \text{Re} \left[ S_{XY}(\omega) \right] \) and \( \text{Re} \left[ S_{YX}(\omega) \right] \) are even functions.

**Proof:** We know that \( S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau \) and also we know that

\[ e^{-j\omega \tau} = \cos \omega t - j \sin \omega t, \quad \text{Re} \left[ S_{XY}(\omega) \right] = \int_{-\infty}^{\infty} R_{XY}(-\tau) \cos \omega t d\tau \]

Since \( \cos \omega t \) is an even function i.e. \( \cos (-\omega t) = \cos (\omega t) \)

\[ \text{Re} \left[ S_{XY}(\omega) \right] = \int_{-\infty}^{\infty} R_{XY}(\tau) \cos \omega t d\tau = \int_{-\infty}^{\infty} R_{XY}(\tau) \cos (-\omega t) d\tau \]

Therefore \( S_{XY}(\omega) = S_{XY}(-\omega) \) Similarly \( S_{YX}(\omega) = S_{YX}(-\omega) \) hence proved.
(3) The imaginary part of $S_{XY}(\omega)$ and imaginary part $S_{YX}(\omega)$ are odd functions of $\omega$ i.e. 

$$\text{Im} \ [S_{XY}(\omega)] \text{ and Im} \ [S_{YX}(\omega)] \text{ are odd functions.}$$

**Proof:** We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau$ and also we know that

$$e^{-j\omega \tau} = \cos \omega \tau - j \sin \omega \tau.$$

$$\text{Im} \ [S_{XY}(\omega)] = \int_{-\infty}^{\infty} R_{XY}(\tau) \sin \omega \tau d\tau = - \int_{-\infty}^{\infty} R_{XY}(\tau) \sin \omega \tau d\tau = - \text{Im} \ [S_{XY}(\omega)].$$

Therefore $\text{Im} \ [S_{XY}(\omega)] = - \text{Im} \ [S_{XY}(\omega)]$ Similarly $\text{Im} \ [S_{YX}(\omega)] = - \text{Im} \ [S_{YX}(\omega)]$ hence proved.

(4) $S_{XY}(\omega)=0$ and $S_{YX}(\omega)=0$ if $X(t)$ and $Y(t)$ are Orthogonal.

**Proof:** From the properties of cross correlation function, we know that the random processes $X(t)$ and $Y(t)$ are said to be orthogonal if their cross correlation function is zero.

$$\text{i.e. } R_{XY}(\tau) = R_{YX}(\tau) = 0.$$

We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau$

Therefore $S_{XY}(\omega)=0$. Similarly $S_{YX}(\omega)=0$ hence proved.

(5) If $X(t)$ and $Y(t)$ are uncorrelated and have mean values and , then

$$S_{XY}(\omega) = 2\pi \overline{X} \overline{Y} \delta(\omega).$$

**Proof:** We know that $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega \tau} d\tau$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} E[X(t)Y(t+\tau)] e^{-j\omega \tau} d\tau$$

Since $X(t)$ and $Y(t)$ are uncorrelated, we know that

$$E[X(t)Y(t+\tau)] = E[X(t)]E[Y(t+\tau)].$$

Therefore $S_{XY}(\omega) = \int_{-\infty}^{\infty} E[X(t)]E[Y(t+\tau)] e^{-j\omega \tau} d\tau$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} \overline{X} \overline{Y} e^{-j\omega \tau} d\tau$$

$$S_{XY}(\omega) = \overline{X} \overline{Y} \int_{-\infty}^{\infty} e^{-j\omega \tau} d\tau$$

Therefore $S_{XY}(\omega)=2\pi \overline{X} \overline{Y} \delta(\omega)$. hence proved.