DIGITAL SIGNAL PROCESSING

C. Ramesh Babu Durai

Published by :
LAXMI PUBLICATIONS (P) LTD
22, Golden House, Daryaganj,
New Delhi-110002.

Phones : **011-23 26 23 68**
011-23 26 23 70

Faxes : **011-23 25 25 72**
011-23 26 22 79

Branches :

- 129/1, IIIrd Main Road, IX Cross Chamrajpet, **Bangalore** (*Phone* : 080-26 61 15 61)
- 26, Damodaran Street, T. Nagar, **Chennai** (*Phone* : 044-24 34 47 26)
- St. Benedict's Road, **Cochin** (*Phone* : 0484-239 70 04)
- Pan Bazar, Rani Bari, **Guwahati** (*Phones* : 0361-254 36 69, 251 38 81)
- 4-2-453, Ist Floor, Ramkote, **Hyderabad** (*Phone* : 040-2475 02 47)
- Adda Tanda Chowk, N.D. 365, **Jalandhar City** (*Phone* : 0181-222 12 72)
- 106/A, Ist Floor, S.N. Banerjee Road, **Kolkata** (*Phones* : 033-22 27 37 73, 22 27 52 47)
- 18, Madan Mohan Malviya Marg, **Lucknow** (*Phone* : 0522-220 95 78)
- 128A, Block 3, First Floor, Noorani Building, L.J. Road, **Mumbai** (*Phone* : 022-24 46 39 98)
- Radha Govind Street, Tharpagna, **Ranchi** (*Phone* : 0651-230 77 64)

EMAIL : colaxmi@hotmail.com

WEBSITE : www.laxmipublications.com

EDS-0652-125-DIGITAL SIGNAL PROCESSING

First Edition

Price : **Rs. 125.00** *Only.*

C—9743/04/12

Laser Typesetted at : Goswami Printers, Delhi-110053 *Printed at* : Ajit Printers, Delhi-110053

Contents

Chapters *Pages*

Chapters | *Pages*

DIGITAL SIGNAL PROCESSING

Unit 1

Chapters :

1. Introduction
2. Applications of Digital Signal Processing

1 Introduction

Characterization and Classification of Signals

Signal

A 'signal' is defined as any physical quantity that varies with time, space and any other independent variable or variables.

More precisely a signal is a function of a set of independent variables. The signal itself carries some kind of information available for observation.

Processing

By 'processing' we mean operating in some fashion on signal to extract some useful information.

Digital

The word 'digital' shall mean that the processing is done with a digital computer or special purpose digital hardware.

Digital Signal Processing

Digital signal processing is concerned with the representation of signals by sequence of numbers or symbols and the processing of these sequence.

The purpose of such processing may be to estimate characteristic parameters or transform a signal into form which is in some sense more desirable.

Application

Bio-medical engineering, acoustics, radar, speech communication, data communication, image processing, nuclear science and many others.

1.1 CLASSIFICATION OF SIGNALS

There are five methods of classifying signals based on different features :

(*a*) *Based on independent variable.*

(*b*) *Depending upon the number of independent variable.*

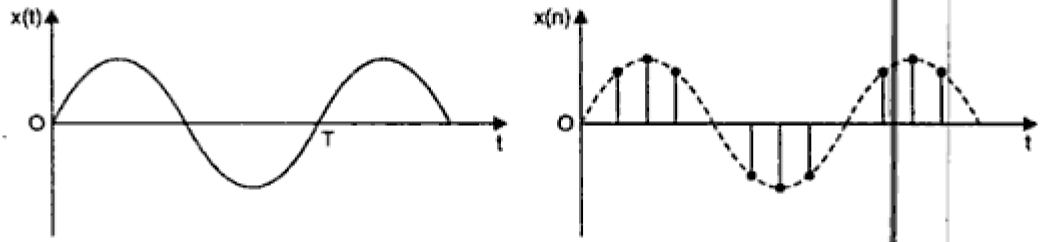
(*c*) *Depending upon the certainity by which the signal can be uniquely described.*

(*d*) *Based on repetition nature.*

(*e*) *Based on reflection.*

(*a*) **Based on independent variable.** Independent variables can be continuous or discrete.

1. *Continuous Time Signal.* It is also referred as analog signal *i.e.*, the signal is represented continuously in time. In simple words, a signal $x(t)$ is said to be a continuous time signal if it is defined for all time.
2. *Discrete Time Signal.* Signals are represented as sequence at discrete time intervals. Thus, the independent variable has discrete values only.



(*a*) Continuous time signal (*b*) Discrete time signal

Fig. 1.1

e.g. Speech signal is an example of analog signal.

A discrete time signal which discrete-valued represented by a finite number of digits is referred to as a "digital signal".

e.g. Digitized music signal stored in CD-ROM disk.

(*b*) **Depending upon the number of independent variable.**

(*i*) *1-D Signals.* It is a function of a single independent variable.

e.g. (*a*) speech signal–independent variable is time.

(*b*) music signal.

(*ii*) *2-D Signal.* It is a function of two independent variables.

e.g. Photographic image signal–two independent variables are the two spatial variables.

Each frame of a black and white video signal is a 2D-image signal that is a function of two discrete spatial variable, with each frame occurring squentially at discrete instants of time.

(*iii*) *M-D Signal.* It is a function of 'M' independent variable in time.

e.g. Video signal.

The black and white video signal can be considered an example of a 3D signal where the three independent variables are two spatial variables and time.

A colour video signal is a three-channel signal composed of three 3-D signals representing the three primary colours : red, green and blue (RGB).

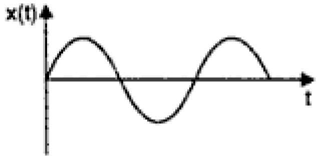
For transmission purpose, the RGB television signal is transformed into another type of 3-channel signal is composed of luminance component and two chrominance components.

(*c*) **Depending upon the certainity by which the signal can be uniquely described as**

(*i*) *Deterministic Signal.* A signal that can be uniquely determined by a well-defined process such as a mathematical expression or rule, or table look-up is called a deterministic signal.

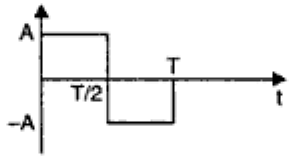
e.g. (*a*) A sinusoidal signal can be represented as,

$$v(t) = V_m \sin \omega t \quad \text{for } t \geq 0.$$



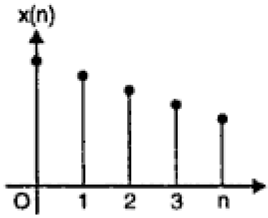
(*b*) A square signal can be defined as

$$x(t) = A \quad \text{for} \quad 0 < t < T/2$$
$$= -A \quad \text{for} \quad T/2 < t < T.$$

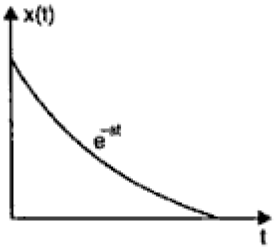


(*c*) An exponential,

discrete time signal $x(n) = e^{an}$ for $n \geq 0$



Continuous time signal $x(t) = e^{-at}$ for $n < 0$.



(*ii*) *Random Signal.* A signal that is generated in a random fashion and cannot be predicted ahead of time is called a "random signal".

e.g. Speech signal, ECG signal, EEG signals.

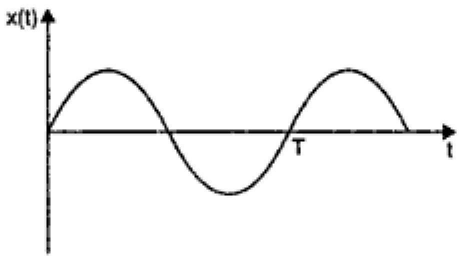
(*d*) **Based on repetition nature.** The signal can be classified into the two types.

(*i*) *Periodic Signal*

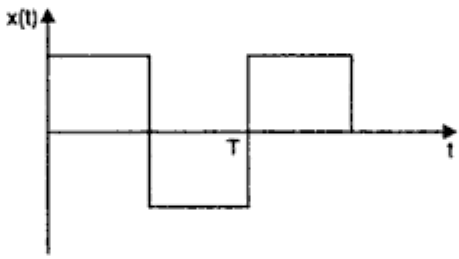
(*a*) *Continuous time.* The periodic Signal means, the signal which repeats every finite, interval of time or a continuous time signal $x(t)$ is a function that satisfies the condition.

$$\boxed{x(t) = x(t + T)} \quad \text{for all '}t\text{'} \qquad \text{...(1.1)}$$

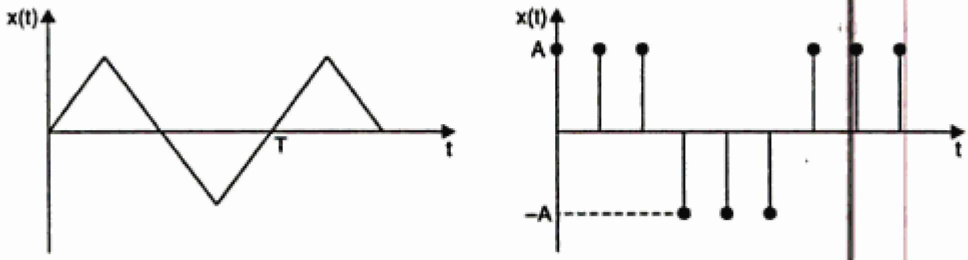
e.g. Sine wave, square wave, triangular wave etc.,



(*a*) Sine wave



(*b*) Square wave



(c) Triangular wave (d) Discrete or impulse wave

Fig. 1.2. Periodic signal.

After every 'T' the signal repeats itself.

The smallest possible value of 'T' for which the equation (1) hold good is known as 'fundamental period', it defines the duration of one complete cycle of $x(t)$.

The reciprocal of fundamental period is known as fundamental frequency of the periodic signal $x(t)$.

i.e., $$f = \frac{1}{T}.$$

$$\Omega = 2\pi f = \frac{2\pi}{T} \quad \Rightarrow \quad \text{Analog angular frequency.}$$

(b) *Discrete time periodic signal.* A discrete time signal is said to be periodic, if it satisfies the relation,

$$\boxed{x(n) = x(n + N)} \quad \text{for all integers of '}n\text{'} \qquad ...(1.2)$$

where, 'N' is a positive integer.

The fundamental period is the smallest positive value of 'N' for which $x(n) = x(n + N)$ hold good. The fundamental angular frequency $\omega = \frac{2\pi}{N}$.

where, ω–discrete angular frequency.

(ii) *Non-periodic Signal.* Any signal $x(t)$ for which there is no value of T to satisfy the relation given in equation $x(t) = x(t + T)$ is known as non-periodic signal.

i.e., $x(t) \neq x(t + T)$, for all 't'

e.g.

CT Signal DT Signal

Fig. 1.3. Non-periodic signal.

(*e*) **Based on reflection.** It can be defined as,

(*i*) *Even signal (symmetric).* A signal $x(t)$ or $x(n)$ is referred to as even signal, if it is identical to its time reversal counter part *i.e.*, with its about the origin.

$$x(t) = x(-t) \quad \text{for all '}t\text{'} \quad \text{CT even signal.}$$

and

$$x(n) = x(-n) \quad \text{for all '}n\text{'} \quad \text{DT even signal.}$$

*Even signals are symmetric with respect to vertical axis.

e.g.

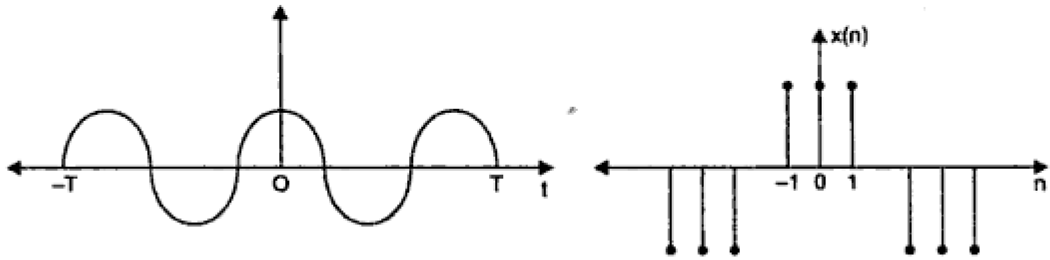


Fig. 1.4. Even signal.

(*ii*) *Odd signal (antisymmetric).* A signal is said to be odd signal if,

$$x(t) = -x(-t) \quad \text{for} \quad \text{CT signal.}$$

$$x(n) = -x(-n) \quad \text{for} \quad \text{DT signal.}$$

e.g.

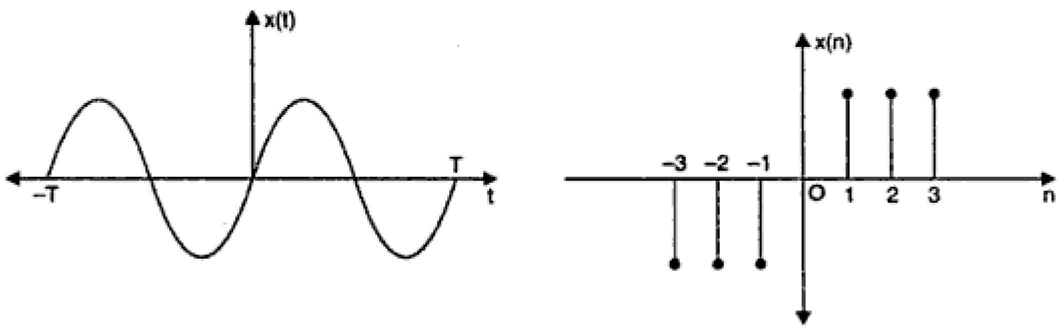


Fig. 1.5. Odd signal.

A signal can be broken into two parts one of which is even and the other is odd.

$$\text{Even }[x(n)] = 1/2 \quad \text{for } n < 0$$
$$= 1 \quad \text{for } n = 0$$
$$= 1/2 \quad \text{for } n > 0$$
$$\text{Odd }[x(n)] = -1/2 \quad \text{for } n < 0$$
$$= 0 \quad \text{for } n = 0$$
$$= 1/2 \quad \text{for } n > 0.$$

Thus,

$$\text{Even }[x(n)] = \frac{1}{2}[x(n) + x(-n)] \qquad ...(1.3)$$

$$\text{Odd }[x(n)] = \frac{1}{2}[x(n) - x(-n)] \qquad ...(1.4)$$

Properties of even and odd signal :

1. The sum of two even signals are even signal.
2. The sum of two odd signals are odd.
3. The sum of an even signal and an odd signal is neither even nor odd signal.
4. The product of two even signal is even.
5. The product of two odd signal is even.
6. The product of even signal and an odd signal is odd.

1.2 MULTI CHANNEL

A signal can be generated by a single source or by multiple sources or multiple sensors. In the former case, it is a (single) scalar signal and in the later case it is a vector signal, often called a multichannel signal.

These type of signals can be represented in vector form as,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{...(1.5)}$$

Equation represents a 3-channel signal.

e.g. In electrocardiography [ECG] for example 3-lead and 12-lead electrocardiographs are often used in practice, which result in 3-channel and 12-channel signals.

1.3 MULTI DIMENSIONAL SIGNALS

If a signal is a function of a single independent variable, then it is called as one-dimensional signal. Similarly, if signal is a function of N-independent variables, it is called as N-dimensional signal.

e.g.

- Picture signal is a two dimensional signal, since the intensity $I(x, y)$ is a function of two independent variables x and y.
- Black and white television picture is an example of 3-dimensional signal because brightness $I(x, y, t)$ is a function of three independent variables x, y and t (time).
- It is also possible to have multichannel and multidimensional signals simultaneously. For example, a colour TV picture is described by three intensity functions of form $I_r(x, y, t)$ [red], $I_g(x, y, t)$ [green], and $I_b(x, y, t)$ [blue].

Hence colour TV picture is a three dimensional and three channel signal, which can be represented by the vector.

$$I(x, y, t) = \begin{bmatrix} I_r(x, y, t) \\ I_g(x, y, t) \\ I_b(x, y, t) \end{bmatrix} \quad \text{...(1.6)}$$

1.4 CONTINUOUS-TIME VERSUS DISCRETE-TIME SIGNALS

(1) Signals can be further classified into different categories depending on the characteristics of the time (independent) variables and the values they take.

Continuous

- Continuous-time signals or analog signals are defined for every value of time and they take on values in the continuous interval (a, b).

where a can be $-\infty$

b can be $+\infty$.

Mathematically, these signals can be described by functions of a continuous variable.

e.g. Speech signals $x_1(t) = \cos \pi t$

$$x_2(t) = e^{-|t|}, \quad -\infty < t < \infty.$$

Discrete

- Discrete time signals are defined only at certain specific values of time. These time instant need not be equidistant, but generally they are taken at equally spaced intervals for convenience.

  e.g.

  $$x(t_n) = e^{-|t_n|}, n = 0, \pm 1, \pm 2 ......$$

  index 'n' of the discrete-time instants as the independent variables.

In applications, discrete-time signal may arise in two ways

- In practical setting, such sequence (n) can often arise from periodic sampling of an analog signal. In this case, the numeric value of the n^{th} number in the sequence is equal to the value of analog signal $x_a(t)$ at time nT *i.e.*,

  $$x(n) = x(nT).$$

  The quantity T is called sampling period and its reciprocal is the sampling frequency.
- By accumulating a variable over a period of time. For example, counting the number of cars in a given street every hour, or recording the value of gold every day, results in discrete-time signals.

(2) **Continuous-valued and Discrete-valued Signals.** A signal is said to be continuous valued signal if it takes on all possible values on a finite or infinite range. On the other hand, if the signal allowed to take on values from the given set, it is said to discrete-valued signal. Normally, these values are equidistance and hence can be expressed as an integer multiple of the distance between two successive values.

If the signal to be processed is in analog form, it is converted to a digital signal by sampling the analog signal at discrete instants in time, obtaining a discrete-time signal, and then by quantizing its values to a set of discrete values.

Quantization. The process of converting a continuous-valued signal into a discrete-valued signal, called quantization.

(3) **Deterministic Versus Random Signals**

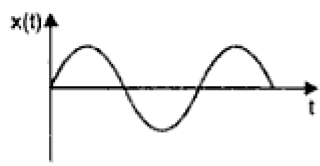
Depending upon the certainity by which the signal can be uniquely described as

(*i*) *Deterministic Signal.* A signal that can be uniquely determined by a well-defined process such as a mathematical expression or rule, or table look-up is called a deterministic signal.

e.g.

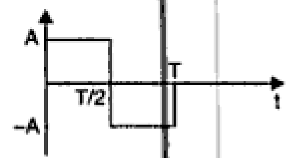
(*a*) A sinusoidal signal can be represented as,

$$v(t) = V_m \sin \omega t \text{ for } t \geq 0.$$



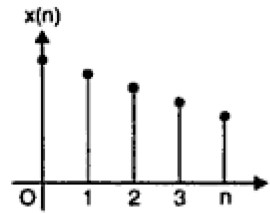
(*b*) A square signal can be defined as

$$x(t) = A \quad \text{for} \quad 0 < t < T/2$$
$$= -A \quad \text{for} \quad T/2 < t < T.$$

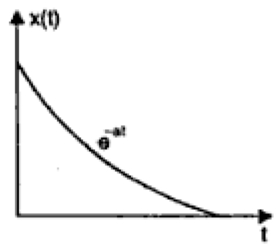


(*c*) An exponential,

discrete time signal $x(n) = e^{an}$ for $n \geq 0$



Continuous time signal $x(t) = e^{-at}$ for $n < 0$.



(*ii*) *Random Signal.* A signal that is generated in a random fashion and cannot be predicted ahead of time is called a "random signal".

e.g. Speech signal, ECG signal, EEG signals.

1.5 FREQUENCY CONCEPT IS CONTINUOUS TIME AND DISCRETE-TIME SIGNALS

We know that the frequency is closely related to a periodic motion which is described by sinusoidal functions. As the frequency is directly related with time $\left(\text{Frequency} = \dfrac{1}{\text{Time period}}\right)$, therefore, we should expect that the nature of time (continuous or discrete) would affect the nature of frequency accordingly.

1.5.1 Continuous Time Sinusoidal Signals

A continuous time signal is mathematically described as,

$$x_a(t) = A\cos(\Omega t + \theta)\,; \quad -\alpha < t < +\alpha \qquad ...(1.7)$$

The above signal is shown in Fig. (1.6)

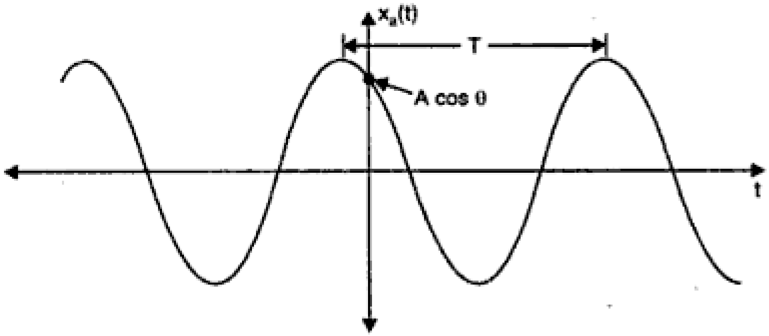


Fig. 1.6. Continuous time sinusoidal signal.

This signal is completely characterized by three parameters :

A is the amplitude of sinusoid

Ω is the frequency in radians per second (r/s) and θ is the phase in radians.

Eqn. 1.7 can be written as,

$$x_a(t) = A\cos(2\pi Ft + \theta)\ ; \qquad -\alpha < t < +\alpha \qquad \text{...(1.8)}$$

where, $\Omega = 2\pi F$.

$F = \frac{1}{T}$ = cycles per second.

T – period of sinusoid.

The analog signal given by eqn. (1.8) has the following properties :

(1) *If* $x_a(t+T) = x_a(t)$, then $x_a(t)$ *is periodic with period T, where* $T = \frac{1}{F}$ *is the fundamental period.*

Proof.

$$x_a(t) = A\cos(2\pi Ft + \theta)$$
$$x_a(t+T) = A\cos[2\pi F(t+T) + \theta]$$
$$x_a(t+T) = A\cos(2\pi ft + \theta)$$
$$x_a(t+T) = x_a(t).$$

(2) *Continuous-time sinusoidals with different (distinct) frequencies are themselves different.*

(3) *Increasing the frequency F results in an increase in the rate of oscillation of the signals.*

Now the sinusoidal signal may be expressed as,

$$x_a(t) = A\cos(\Omega t + \theta)$$
$$= \frac{A}{2}e^{j(\Omega t+\theta)} + \frac{A}{2}e^{-j(\Omega t+\theta)}.$$
$$(\because \quad e^{\pm j\phi} = \cos\phi \pm j\sin\phi)$$

Note that, a sinusoidal signal can be obtained by adding two-equal amplitude complex conjugate exponential signal, sometimes called 'phasors'.

Respresentation of a cosine function by a pair of complex-conjugate exponential.

As time progresses the phasors rotate in opposite directions with angular frequencies ± Ω radians/second.

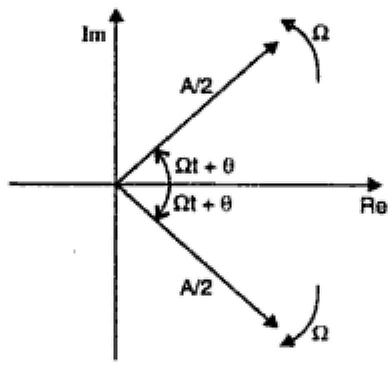


Fig. 1.7. Representation of cosine function.

Since, a ' positive frequency' corresponds to counter clockwise uniform angular motion, a 'negative frequency' simply corresponds to clockwise angular motion. The frequency range for analog sinusoids is $-\infty < F < \infty$.

1.5.2 Discrete-time Sinusoidal Signals

Mathematically, a discrete-time sinusoidal signal is represented as,

$$x(n) = A\cos(\omega n + \theta) \qquad -\alpha < n < \alpha$$

$$x(n) = A\cos(2\pi fn + \theta) \quad -\alpha < n < \alpha$$

where, n is an integer value, called the sample number

A is the amplitude

ω is the frequency is radians per sample.

and θ is the phase in radians.

Fig. 1.8 represents a discrete-time sinusoid with $\omega = \pi/3$ radians per sample [$2\pi f = \pi/2$ or $f = 1/12$ cycle per sample] and $\theta = 0$.

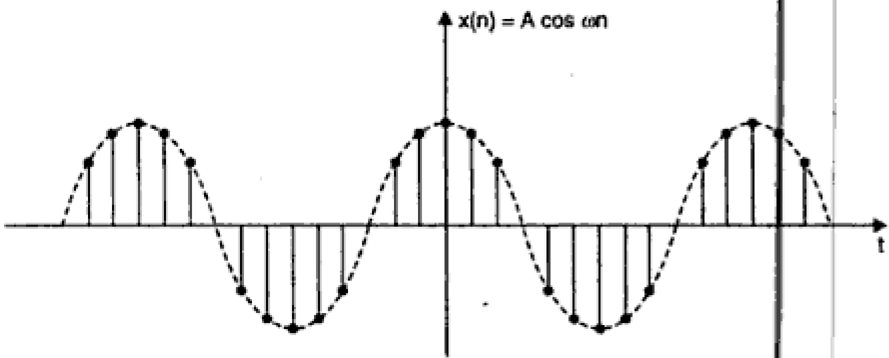


Fig. 1.8. Discrete time sinusoid signal.

The discrete-time signals are characterised by the following properties :

(1) *Discrete-time sinusoids are periodic only if its frequency f_0 is a rational number.*

* The signal $x(n)$ is periodic with period N(N > 0) if and only if

$$x(n + N) = x(n) \qquad ...(1.9)$$

The smallest value of N is called the fundamental period.

Proof : For a sinusoid with frequency f_0 to be periodic, we should have,

$$x(n + N) = \cos[2\pi f_0 (N + n) + \theta]$$
$$= \cos[2\pi f_0 n + \theta]$$

The above relation is true if and only if, there exists an integer k such that,

$$2\pi f_0 N = 2\pi k.$$

i.e., $\therefore$

$$\boxed{f_0 = \frac{k}{N}} \qquad ...(1.10)$$

Therefore, the discrete-time sinusoids are periodic only if its frequency f_0 can be expressed as rational number (ratio of two integers).

To determine the fundamental period N of a periodic sinusoid, k and N is eqn. 1.10 should be relatively prime. Then the fundamental period of sinusoid is equal to N. For example, *i.e.,*

$$f_1 = \frac{21}{40}.$$

then fundamental period N_1 is 40 and if

$$f_2 = \frac{20}{40} = \frac{1}{2}$$

then fundamental period N_2 is 2. We observe that a small change in frequency may result in a large change in the period.

(2) *Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.*

Proof : Consider a sinusoid $\cos(\omega_0 n + \theta)$. If the frequencies are separated by 2π, then,

$$\cos[(\omega_0 + 2\pi)n + \theta] = \cos[\omega_0 n + 2\pi n + \theta]$$
$$= \cos[\omega_0 n + \theta].$$

Therefore, all the sinusoid signals,

$$x_k(n) = A\cos(\omega_k n + \theta)\,; \quad k = 0, 1, 2 \,........$$

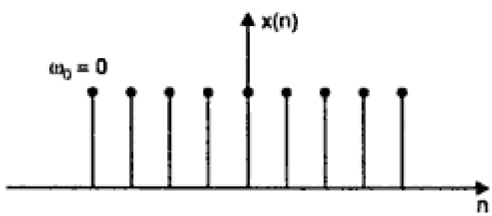
where, $w_k = w_0 + 2\pi k$, are identical (distinguishable).

Conclusion. The discrete-time sinusoids with frequencies $|\omega| \le \pi$ or $|f| \le 1/2$ are unique. The sequence resulting from a sinusoid with frequency $|\omega| > \pi$ or $|f| > 1/2$, are identical to the sequence obtained from the sinusoid with frequency $|\omega| < \pi$ or $|f| < 1/2$.

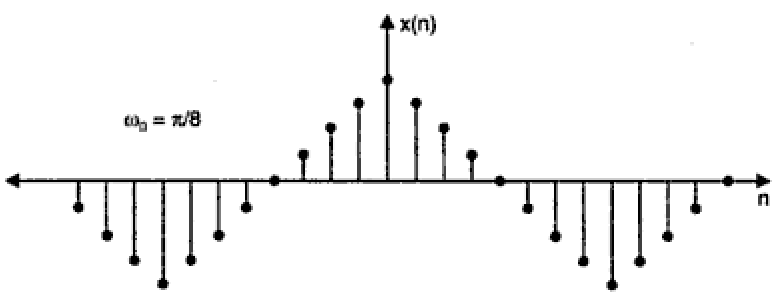
(3) *In discrete-time sinusoids, highest rate of oscillations is attained when $\omega = \pi$ (or $-\pi$) or equivalently $f = 1/2$ (or $-1/2$).*

To investigate the characteristic of the sinusoids, let us vary the frequency from 0 to π.

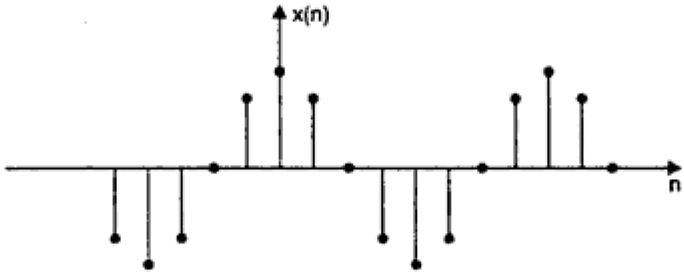
(*i*) $\omega_0 = 0$; d_c *i.e.*, no oscillations ; $N = \alpha$



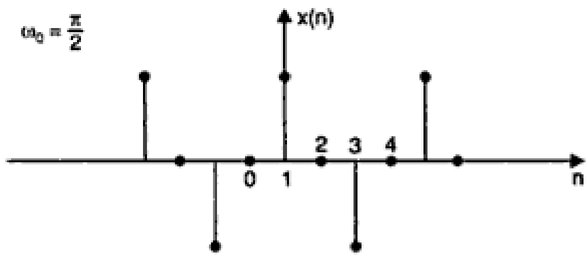
(*ii*) $\omega_0 = \pi/g$, or $f_0 = 1/16$; 16 samples in one cycle ; N = 16.



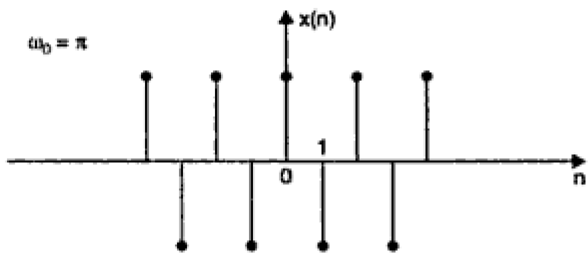
(*iii*) $\omega_0 = \pi/4$ or $f_0 = \frac{1}{8}$; 8 samples in one cycle ; N = 8.



(*iv*) $\omega_0 = \pi/2$ or $f_0 = 1/4$; 4 samples in one cycle ; N = 4.



(*v*) $\omega_0 = \pi$ or $f_0 = \frac{1}{2}$; 2-samples in one cycle ; N = 2.



1.5.3 Harmonically Related Complex Exponentials

Sinusoidal signals and complex exponentials play a major role in the analysis of signals and systems. In some cases we deal with sets of harmonically related complex exponentials (or sinusoidal). These are sets of periodic complex exponentials with fundamental frequencies that are multiples of a single positive frequency.

Continuous-time exponentials. The basic signals for continuous-time harmonically related exponentials are,

$$S_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t} \;;\; k = 0, \pm 1, \pm 2 \;..... \qquad ...(1.11)$$

we note that for each value of k, $S_k(t)$ is periodic with fundamental period $\frac{1}{kF_0} = \frac{T_p}{k}$ or fundamental frequency kF_0.

Since a signal that is periodic with period T_p/k is also periodic with period $k\,[T_p/k] = T_p$ for any positive integer k, we see that all of the $S_k(t)$ have a common period of T_p.

From the basic signals in eqn. (1.11), we can construct a linear combination of harmonically related complex exponentials of the form,

$$x_a(t) = \sum_{k=-\infty}^{\infty} C_k S_k(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\Omega_0 t} \quad ...(1.12)$$

where, C_k, $k = 0, \pm, 1, \pm 2$ .... are arbitrary complex constants.

The signal $x_a(t)$ is periodic with fundamental period $T_p = 1/F_0$ and its representation in terms of eqn. (1.12) is called the Fourier series expansion for $x_a(t)$.

Discrete-Time exponentials

Since a discrete-time complex exponential is periodic if its relative frequency is a rational number, we choose $f_0 = 1/N$ and we define the sets of harmonically related complex exponentials by

$$S_k(n) = e^{j2\pi k f_0 n} \quad ; k = 0, \pm 1, \pm 2, \pm 3 ...... \quad ...(1.13)$$

In contrast to the continuous-time case, we note that,

$$S_{K+N}(n) = e^{j2\pi n(k+N)/N} = e^{j2\pi n} S_k(n) = S_k(n).$$

This means that, consistent with eqn. (1.9) [*i.e.*, $x(N + n) = x(n)$] there are only N distinct periodic complex exponential is the set described by eqn. (1.13).

Furthermore, all members of the set have a common period of N samples.

Clearly, we can choose any consecutive N complex exponentials say from $k = n_0$ to $k = n_0 + N - 1$ to form a harmonically related set with fundamental frequency $f_0 = 1/N$.

For our convenience, we choose the set that corresponds to $n_0 = 0$, that is the set,

$$S_k(n) = e^{j2\pi kn/N} ; k = 0, 1, 2 ..... N - 1.$$

As in the case of continuous-time signals, it is obvious that the linear combination,

$$x(n) = \sum_{k=0}^{N-1} C_k S_k(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N},$$

results in a periodic signal with fundamental period 'N'. The sequence of $S_k(n)$ is called the k^{th} harmonic of $x(n)$.

1.6 ENERGY AND POWER SIGNALS (CONTINUOUS TIME-INSTANTS)

Signals can also be classified as those having finite energy or finite average power. However, there are some signals which can neither be classified as energy signals nor power signals. Consider, a voltage source $v(t)$, across a unit resistance R, conducting, a current $i(t)$. The instantaneous power dissipated by the resistor is

$$P(t) = v(t)\, i(t) = \frac{v^2(t)}{R} = i^2(t)\, R.$$

Since $R = 1\,\Omega$, we have,

$$P(t) = v^2(t) = i^2(t).$$

The total energy and the average power are defined as the limits.

$$E = \lim_{T \to \infty} \int_{-T}^{T} i^2(t)\, dt \text{ joules} \quad ...(1.14)$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} i^2(t)\, dt \text{ watts} \quad ...(1.15)$$

The total energy and the average power normalised to unit resistance of any arbitrary signal $x(t)$ can be defined as,

$$E = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt \text{ Joules.} \quad \ldots(1.16)$$

and

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \text{ watts} \quad \ldots(1.17)$$

} Continuous

The energy signal is one which has finite energy and zero average power, *i.e.*, $x(t)$ is an energy signal if $0 < E < \infty$ and $P = 0$.

The power signal is one which has finite average power and infinite energy, *i.e.*, $0 < P < \infty$ and $E = \infty$.

If the signal does not satisfy any of these two conditions, then it is neither an energy nor a power signal.

1.7 SINGULARITY FUNCTIONS

Singularity functions are an important classification of non-periodic signals. They can be used to represents more complicated signals.

The unit impulse function, sometimes referred to as "delta function", is the basic singularity function and all other singularity functions can be derived by repeated integration or differentiation of the delta function. The other commonly used singularity functions are the unit-step and unit-ramp functions.

1.7.1 Unit-Impulse Function

The unit-impulse function is defined as,

$$\delta(t) = 0, \quad t \neq 0 \quad \ldots(1.18)$$

and

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \quad \ldots(1.19)$$

The eqn. (1.18) and (1.19) indicate that the area of the impulse function is unity and this area is confined to an infinitesimal interval on the t-axis and concentrated at $t = 0$.

The unit impulse function is very useful in continuous-time analysis. It is used to generate the system response providing fundamental information about the system characteristics.

In discrete-time domain, the unit-impulse signal is called a "unit-sample signal".

It is defined as, $\delta(n) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases}$.

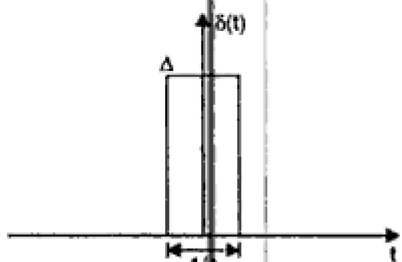


Fig. 1.9 (*a*). Continuous time impulse signal.

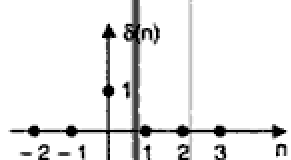


Fig. 1.9 (*b*). Discrete time impulse signal.

1.7.2 Unit-Step Function

The integral of the impulse function $\delta(t)$ gives,

$$\int_{-\infty}^{t} \delta(t) \, dt = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases}$$

Since, the area of the impulse function is all concentrated at $t = 0$, for any value of $t < 0$, the integral becomes zero and for $t > 0$, $\int_{-\infty}^{t} \delta(t)\,dt = 1$.

The integral of the impulse function is also a singularity function and called the unit-step function and is represented as,

$$u(t) = \begin{cases} 0 & , \ t < 0 \\ 1 & , \ t > 0 \end{cases}$$

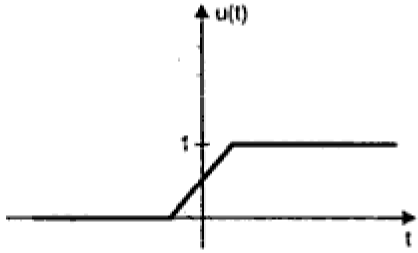


Fig. 1.10 (*a*). Continuous time unit step signal.

The value at $t = 0$ is taken to be finite and in most cases it is unspecified. The discrete-time unit-step signal is defined as

$$u(n) = \begin{cases} 0 & , \ n < 0 \\ 1 & , \ n \geq 0 \end{cases}.$$

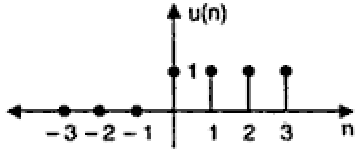


Fig. 1.10 (*b*). Discrete time unit step signal.

1.7.3 Unit-Ramp Function

The unit-ramp function, $r(t)$ can be obtained by integrating the unit-impulse function twice or integrating the unit-step function once,

i.e.,
$$r(t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \delta(\tau)\,d\tau\,d\alpha.$$
$$= \int_{-\infty}^{t} u(\alpha)\,d\alpha.$$

$$\boxed{r(t) = \int_{0^+}^{t} 1\,d\alpha}$$

That is,
$$r(t) = \begin{cases} 0 & , \ t < 0 \\ t & , \ t > 0 \end{cases}.$$

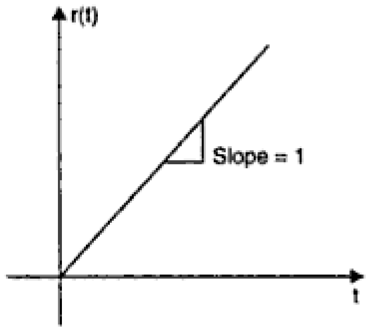


Fig. 1.11 (*a*). Continuous time ramp signal.

A ramp signal starts at $t = 0$ and increases linearly with time 't'.

In discrete-time domain, the unit-ramp signal is defined as,

$$r(n) = \begin{cases} 0 & , \ n < 0 \\ n & , \ n > 0 \end{cases}.$$

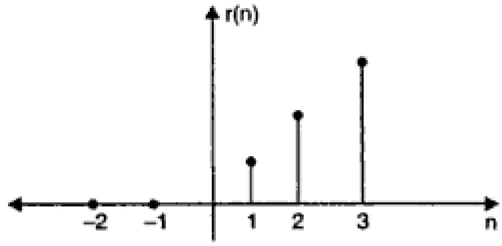


Fig. 1.11 (*b*). Discrete time ramp signal.

1.7.4 Unit-Pulse Function

An unit-pulse function, $\pi(t)$, is obtained from unit-step signal as shown below.

$$\pi(t) = u(t + 1/2) - u(t - 1/2)$$

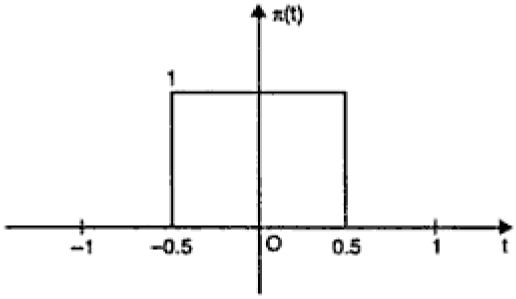


Fig. 1.12. Unit pulse signal.

The signal $u(t + 1/2)$ and $u(t - 1/2)$ are the unit-step signals shifted by 1/2 units in the time axis towards the left and right respectively.

Advantage. The advantage of the singularity function is that any arbitrary signal that is made up of straight line segments can be represented in terms of step and ramp functions.

Properties of δ(t)

(1) $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

(2) $\int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0)$

Proof for (2) :

$$\int_{-\infty}^{\infty} x(t) \lim_{\Delta \to 0} \delta_{\Delta}(t)\, dt \qquad \because \ \delta(t) = \lim_{\Delta \to 0} \delta_{\Delta}(t)$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{-\infty}^{\infty} x(t)\, P_T(t) dt \qquad = \lim_{\Delta \to 0} \frac{1}{\Delta} P_T(t) = \frac{1}{\Delta} \lim_{\Delta \to 0} P_T(t)$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{-\infty}^{\infty} x(t)\, dt = \mathbf{x(0)}.$$

According to pulse function property,

$$P_T(t) = 1$$

(3) $\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$

(4) $\int_{-\infty}^{\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$

(5) $\delta(at) = \frac{1}{|a|} \delta(t)$

(6) $x(t)\, \delta(t - t_0) = x(t_0)$

(7) $x(t_0)\, \delta(t - t_0) = x(t_0)$

(8) $\int_{t_1}^{t_2} x(t) \delta^n (t - t_0)\, dt = (-1)^n x^n (t_0)$.

Proof for (8): $\frac{d}{dt} [x(t)\, \delta(t - t_0)] = x(t)\, \dot{\delta}\, (t - t_0) + \dot{x}\, (t)\, \delta(t - t_0)$

$$= x(t) \dot{\delta}(t - t_0) + \dot{x}(t_0)\, \delta(t - t_0),\ t_1 < t_0 < t_2$$

Integrating, we get

$$\int_{t_1}^{t_2} \frac{d}{dt}[x(t)\,\delta(t-t_0)]\,dt = \int_{t_1}^{t_2} [x(t)\,\dot{\delta}(t-t_0)]\,dt + \int_{t_1}^{t_2} [\dot{x}(t_0)\delta(t-t_0)]\,dt$$

$$\Big[x(t)\,\delta(t-t_0)\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0)$$

L.H.S. = 0.

Therefore, $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt + \dot{x}(t_0) = 0$

i.e., $\int_{t_1}^{t_2} x(t)\,\dot{\delta}(t-t_0)\,dt = -\,\dot{x}(t_0)$.

Similarly, $\int_{t_1}^{t_2} x(t)\,\ddot{\delta}(t-t_0)\,dt = \ddot{x}(t_0)$

Hence, $\int_{t_1}^{t_2} x(t)\,\delta^n(t-t_0)\,dt = (-1)^n x^n(t_0)$.

1.8 ENERGY SIGNALS AND POWER SIGNALS (DISCRETE-TIME INSTANTS)

The energy of a signal $x(n)$ is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2.$$

The energy of a signal can be finite or infinite. If E is finite, then $x(n)$ is an energy signal. Many signal that posses infinite energy, have a finite power. The average power of a discrete-time signal $x(n)$ is defined as,

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2.$$

If we define the signal energy of $x(n)$ over the finite interval $-N \le n$:

$$E_N = \sum_{n=-N}^{N} |x(n)|^2.$$

then we can express the signal energy E as,

$$E = \lim_{N\to\infty} E_N$$

and the average power of the signal $x(n)$ as,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} E_N.$$

Clearly, if E_N is finite, P = 0

on the other hand, if E_N is infinite, the power P may be either finite or infinite.

If P is finite (and non-zero), the signal is called a "power signal".

Problem 1. *Determine the power and energy of the unit-step sequence.*

Sol. The average power of the unit-step signal is,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} \boxed{u^2(n)}$$

$$P = \lim_{N \to \infty} \frac{1+N}{2N+1} = \lim_{N \to \infty} \frac{1/N+1}{2+(1/N)} \qquad \boxed{P = 1/2.}$$

Problem 2. *Determine which of the following signals are periodic.*

(*a*) $x_1(t) = \sin 15\,\pi t$ (*b*) $x_2(t) = \sin 20\,\pi t$

(*c*) $x_3(t) = \sin \sqrt{2}\,\pi t$ (*d*) $x_4(t) = \sin 5\pi\, t$

(*e*) $x_5(t) = x_1(t) + x_2(t)$ (*f*) $x_6(t) = x_2(t) + x_4(t)$.

Sol. (*a*) $x_1(t) = \sin 15\pi t$ is periodic,

The fundamental period is,

$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{15\pi} = 0.1333 \text{ sec.}$$

(*b*) $x_2(t) = \sin 20\,\pi t$ is periodic,

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{20\pi} = 0.1$ sec.

(*c*) $x_3(t) = \sin \sqrt{2}\pi t$ is period,

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\sqrt{2}\pi} = 1.41421$ sec.

(*d*) $x_4(t) = \sin 5\pi t$ is periodic

The fundamental period is, $T_0 = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{5\pi} = 0.4$ sec.

(*e*) $x_5(t) = x_1(t) + x_2(t)$.

The fundamental period of $x_1(t) = T_{01} = 0.133$ sec. and the fundamental period of $x_2(t) = T_{02} = 0.1$ sec.

The ratio of fundamental frequencies,

$$\frac{T_{01}}{T_{02}} = \frac{0.1333}{0.1}$$, cannot be expressed as a ratio of integers.

Hence, $x_5(t)$ is not periodic.

(*f*) $x_6(t) = x_2(t) + x_4(t)$.

The fundamental period of $x_2(t) = T_{02} = 0.1$ sec and the fundamental period of $x_4(t) = T_{04} = 0.4$ sec.

The ratio of fundamental frequencies,

$\dfrac{T_{02}}{T_{04}} = \dfrac{0.1}{0.4} = \dfrac{1}{4}$, can be expressed as ratio of integers. Hence, $x_6(t)$ is periodic.

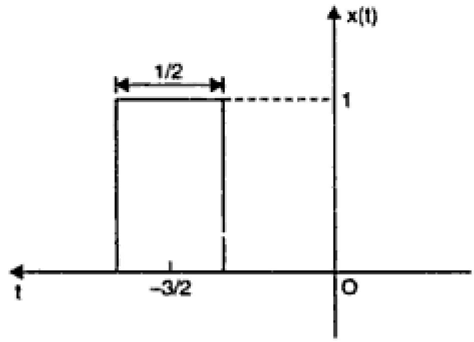
Problem 3. *Sketch of the following signal :*

(*a*) $x(t) = \pi(2t + 3)$ (*b*) $x(t) = 2\pi(t - 1/4)$

(*c*) $x(t) = \cos(20\,\pi t - 5\pi)$ and (*d*) $x(t) = r(-0.5t + 2)$.

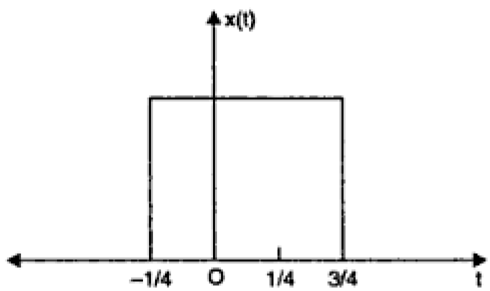
Sol. (*a*) $x(t) = \pi(2t + 3) = \pi[2\,(t + 3/2)]$

Here, the signal is shifted to left, with centre at – 3/2. Since $a = 2$ *i.e.*, $|a| > 1$, the signal is compressed. The signal width becomes 1/2 with unity amplitude.



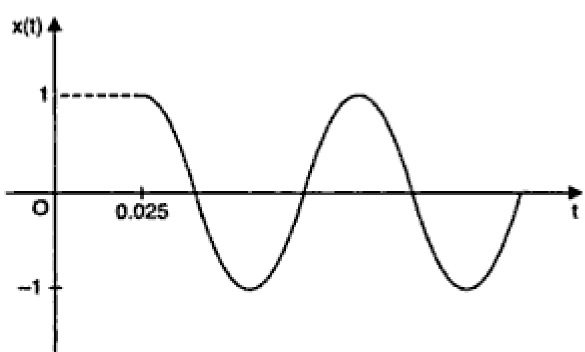
(*b*) $x(t) = 2\pi(t - 1/4)$.

Here the signal is shifted to the right, with centre at 1/4. Since $a = 1$, the signal width is 1 and amplitude is 2.



(*c*) $x(t) = \cos\,(20\,\pi t - 5\pi) = \cos\,[2\pi\,(t - 1/4)]$

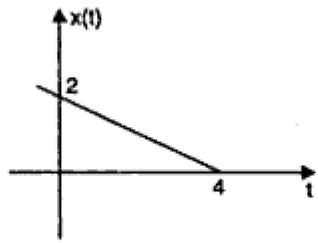
Here the signal $x(t)$ is shifted by quarter cycle to the right.



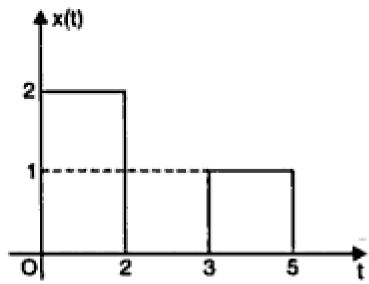
(*d*) $x(t) = r(-\,0.5t + 2) = r\left[-\,0.5\left(t - \dfrac{2}{0.5}\right)\right] = r\left[-\,0.5\,(t - 4)\right]$

The given ramp signal is reflected through the origin and shifted to the right at $t = 4$.

The signal is expanded by $\frac{1}{0.5} = 2$. When $t = 0$, the magnitude of the signal $x(t) = 2$.



Problem 4. *Write down the corresponding equation for the given signal,*



Sol. Representation through addition of two unit step functions, the signal $x(t)$ can be obtained by adding both the pulses, *i.e.*,

$$x(t) = 2[u(t) - u(t-2)] + [u(t-3) - u(t-5)]$$

Representation through multiplication of two unit step functions,

$$x(t) = 2[u(t)\, u(-t+2)] + [u(t-3)\, u(-t+5)]$$
$$= 2[u(t)\, u(2-t)] + [u(t-3)\, u(5-t)]$$

Problem 5. *Plot the following signals for the given $x(n) = (5-n)\,[u(n) - u(n-5)]$*

(i) $y_1(n) = x(4-n)$ *(ii)* $y_2(n) = (2n-3)$.

Sol. The given $x(n) = (5-n)\,[u(n) - u(n-5)]$ is plotted as shown below,

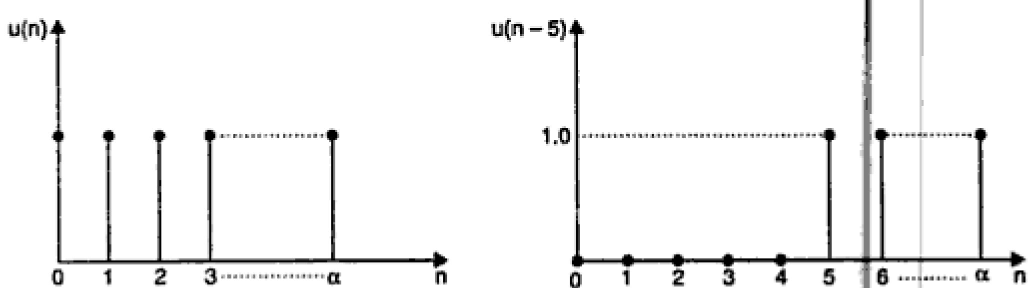


Fig. (*a*) Fig. (*b*)

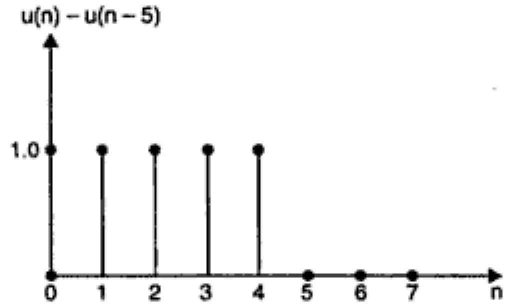


Fig. (c)

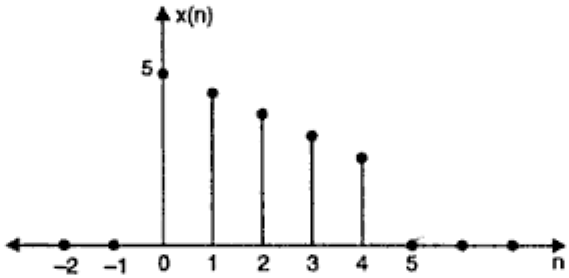


Fig. (d)

Now Fig. (c) is multiplied by scale factor $(5 - n)$ thus we get wave Fig. (d).

(i) $y(n) = x(4 - n)$

where, $$x(n) = (5 - n)\,[u(n) - u(n - 5)]$$

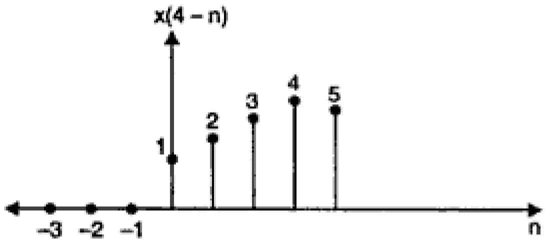
$$x(4 - n) = (5 - 4 + n)\,[u(4 - n) - u(4 - n - 5)]$$

$$= (1 + n)\,[u(4 - n) - u(-1 - n)]$$

$u\,(4 - n)$ means the sequence will exist between $-\infty < n < 4$.

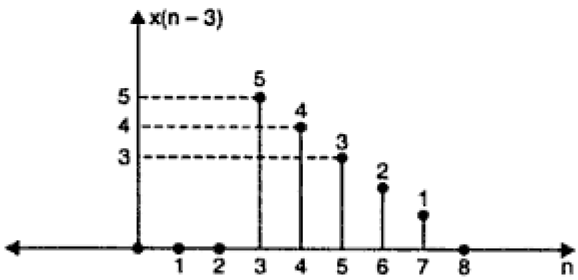
$u\,(-1 - n)$ means sequence will exist between $-\infty < n \le -1$.

When subtracting these two sequence the resultant sequence existing between $0 \le n \le 4$ and it is plotted as shown below :

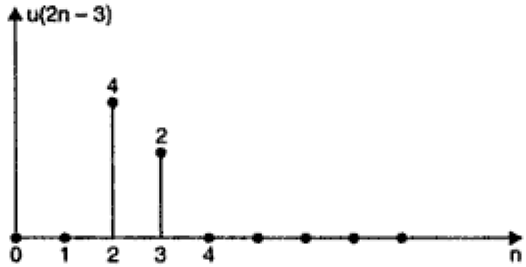


(ii) $y_2\,(n) = (2n - 3)$.

In this case, $y = x(n - 3)$ is plotted as shown below :



and its compressed version of $y(n) = x(2n - 3)$ is plotted as shown below :



1.9 SIGNAL PROCESSING

A system may also be defined as a physical device that performs an operation on a signal.

For example, a filter used to reduce the noise and interference corrupting a desired information bearing signal is called a system.

When we pass a signal through a system, as in filtering, we say that we have processed the signal. In this case, the processing of the signal involves filtering the noise and interference from the desired signal.

Let us see a very simple signal processing example,

To explore the parallels and divergence between digital and analog filtering design methodology, a very simple problem will be approached by both methods.

Assume that a low frequency signal $s(t)$ band limited to f_s(Hz), is observed in an additive noisy environment to give a received signal $x(t)$ given by,

$$x(t) = s(t) + n(t) \qquad ...(1.20)$$

The noisy signal $n(t)$ is assumed to be band limited while noise, that is, its spectral content has equal per unit band width power for dc to f_n (Hz). The problem is to operate on $x(t)$ in some way to obtain an estimate $\hat{S}(t)$ of the signal $S(t)$. Two solutions are presented, the first being an analog signal processor, the second a digital signal processor. Both processors will be based on a direct approach of obtaining better estimates of $S(t)$ by using a low pass filter to minimize the effect of the high frequency component of the interfacing noise.

(*a*) **Analog signal processing.** The analog processor will be assumed to take the general form shown in Fig. (1.13)

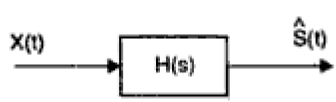


Fig. 1.13. Analog signal processor.

Using the given requirements above, classical techniques for filter design can be used to establish a transfer function $H(s)$ through a Butterworth or chebyshev low pass filter approximation. From $H(s)$ a linear time invariant circuit could be synthesized ; for example, a simple pole filter consisting a resistor and capacitor as shown in Fig. 1.14 must be suggested.

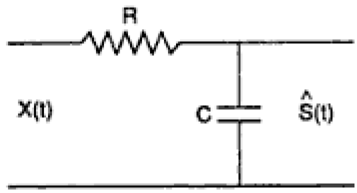


Fig. 1.14. Simple pole filter.

Digital Signal Processing

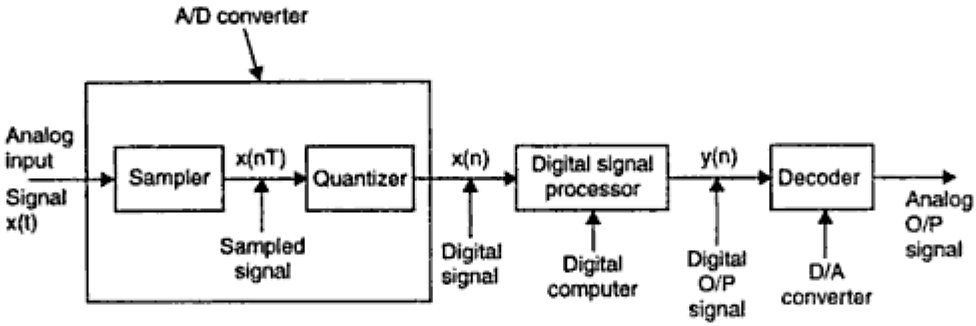


Fig. 1.15. Digital signal processing system.

Following are the main elements of digital signal processing system.

1. Sampler
2. Quantizer
3. Digital Signal Processor
4. Decoder (D/A converter).

A continuous time signal, when sampled at regular intervals is converted into the discrete-time signals by means of sampler. The output of the sampler consists of a sequence of sample values of original analog signal. Note that amplitudes of the sampled signals are not restricted, in principles, any amplitude is permissible.

However if the sampled signal is to be processed in digital computer, its values must be represented by a certain number of bits, so only finite amplitude level is possible. This results in quantization of amplitude which is done by quantizer. The quantized discrete signal $x(n)$ is called a digital signal. This signal is applied to the input of the digital signal processor.

The digital signal processor may be a large programmable digital computer to perform the desired operations on the input signal. It may also be hardwired digital processor configured to perform a specific set of operations such as filtering frequency analysis and so on.

In some applications where the digital output is to be given in analog form, such as speech signal, we must convert the digital signal into the analog signal. Such a operation is performed by Digital to Analog converter (D/A converter).

However, there are other applications, involving signal analysis, where the desired information is conveyed in digital form, therefore, no D/A converter is required. For example, in radar applications, the information extracted from the signal, such as its speed and position of aircraft, may simply be printed on the paper.

1.10 ANALOG VERSUS DIGITAL SIGNAL PROCESSING

Advantages :

(1) **Flexibility.** Digital signal processing operations are flexible as the operations can be changed by changing the program.

(2) **Tolerance.** Unlike analog circuits the operation of the digital circuits does not depend on precise values of the digital signals. As a result, the digital circuits are less sensitive to tolerance component values.

(3) **Component drift with temperature and time.** Digital systems are fairly independent of temperature, aging (time), and most other external parameters, For example, due to change in temperature, the internal resistance R may change in analog systems. On the other hand, digital systems use logic 1 or logic 0 which are independent of temperature.

(4) **System Size.** Analog systems normally use L, C and R, therefore size of hardware is large as compared to digital system.

(5) **Storage.** Digital signals are easily stored on magnetic media (*e.g.* tape and disc) without deterioration or loss of signal fidelity, therefore the signal becomes transportable and can be processed off-line in a remote laboratory. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

(6) **Implementation.** It is very difficult to perform precise mathematical operations on signal in analog form but these same operations can be routinely implemented on the digital computer using hardware.

(7) **Cost.** Digital signal processing allows the sharing of a given processor among a number of signals by time sharing. Thus reducing the cost of processing per signal. This is done by "time-division multiplexing".

Disadvantages :

(1) **System Complexity.** Digital signal processing of analog signals is more complex because of the need for additional pre-and post processing devices such as A/D and D/A converters and their associated filters.

(2) **Band Width.** The second disadvantage associated with digital signal processing is the limited range of frequencies available for processing. This property limits its application particularly in the DSP of analog signals. The signals having extremely wide bandwidth require fast sampling rate A/D coverters. Hence, there are many analog signals with large bandwidth for which the digital signal processing approach is beyond the state of the art of the digital hardware.

(3) **Power.** The another disadvantage of DSP is that signal systems are constructed using active devices (transistor) that consumes power. On the other hand, a variety of analog processing algorithm can be implemented using passive circuits employing inductor, capacitor and resistor that do not need any power. Also active devices are less reliable than passive devices.

REVIEW QUESTIONS

1. Write the major classification of signals.
2. Explain the difference between deterministic signal and random signal with suitable example.
3. Define periodic and aperiodic signals with the help of examples.
4. Explain even and odd signals with the help of examples.
5. Explain energy and power signal with the help of examples.
6. Define the following elementary signals
   (1) unit impulse signal. (2) unit step signal.
   (3) unit ramp signal.
7. Explain the following manipulations for independent variable of a signal
   (1) Time shifting (2) Time scaling
   (3) Time inversion or folding.
8. What are the advantages of Digital signal processing compared to Analog signal processing.
9. Briefly explain multichannel and multidimensional signals.
10. Define continuous time exponential and discrete time exponential signal.
11. Write the properties of impulse response signals.

EXERCISES

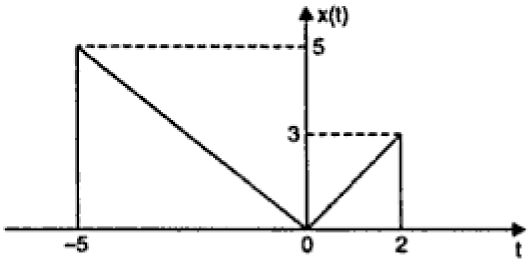
1. Determine which of the following signals are periodic and determine the fundamental period also.
   (1) $x(t) = 20 \sin 25\pi t$ (2) $x(t) = 20 \sin \sqrt{5}\pi t$
   (3) $x(t) = 10 \cos 10\pi t$ (4) $x(t) = 3 \cos (5t + \pi/6)$
   (5) $x(n) = 3 \cos (5n + \pi/6)$ (6) $x(n) = 2 \exp (j(n/6 - \pi))$
   (7) $x(n) = \cos (n/8) \cos (\pi n/8)$.
2. Determine the even and odd components of each of the following signals :
   (1) $x(t) = \cos t + \sin t + \sin t \cos t$ (2) $x(t) = 1 + t + 2t^2 + 5t^3 + 8t^4$.
3. Consider the sinusoidal signal
   $$x(t) = A \cos (\omega t + \theta)$$
   Determine the average power of $x(t)$.
4. Sketch the waveforms of following signals :
   (1) $x(t) = u(t) - u(t-2)$ (2) $x(t) = u(t+1) - 2u(t) + u(t-1)$
   (3) $x(t) = r(t+1) - r(t) + r(t-2)$
5. Given a sinusoidal signal
   $$x(n) = 20 \cos \left[\frac{4\pi}{31} n + \frac{\pi}{5}\right]$$
   Determine the fundamental period of $x(n)$.
6. Given a complex valued exponential signal
   $$x(t) = Ae^{\alpha t + j\omega t} \text{ for } \alpha > 0$$
   Evaluate the real and imaginary components of $x(t)$.

7. Determine the power and rms value for each of the following signals :

(1) $20 \cos\left[100t + \frac{\pi}{3}\right]$ (2) $20 \sin 5t \cos 10t$

(3) $10 \cos 5t \cos 10\, t$ (4) $e^{j\omega t} \cos \omega_0 t$.

8. Figure below shows a signal $x(t)$. For this signal sketch.



(1) $x(t - 4)$ (2) $x(t/10)$

(3) $x(3t - 2)$ (4) $x(3 - t)$.

2 Applications of Digital Signal Processing

2.1 INTRODUCTION

Because of the availability of high resolution spectral analysis, DSP has various application areas, which requires high speed processors to implement the FFT algorithm. It is also popular due to availability of custom made DSP chip which is highly reliable. Speech processing, Audio processing, Radar signal processing and Image processing would be discussed in this chapter.

2.2 APPLICATION TO SPEECH PROCESSING

The signals of speech are one dimensional. DSP is applied to a wide range of problem in speech such as channel vocoders, spectrum analysis etc.

Problems in speech processing can generally be divided into three classes, first is the speech analysis. The speech analysis is performed to extract some desirable information of speech. This system starts with analysis of speech waveform and the desired result is used for speech recognization and speaker indentification. Second type of problem is speech synthesis. In it, input is in written text form and the output is a speech signal. For example, an automatic reading machine for which the input is written text and the output is speech. Finally the third type is speech compression which involves speech analysis followed by speech synthesis. If the speech is transmitted by simply sampling and digitizing, the data rate required is in the order of 90,000 bits per second of speech. Through the use of appropriate coding this can be reduced by factor of 50, depending on the type of system used.

2.2.1 Vocal Mechanism

Production of speech. The two important part responsible for human speech are (*a*) vocal cord and (*b*) vocal tract.

(*a*) *Vocal cord.* It has two bands of tough, elastic tissue, which is located at the opening of the larynx. It vibrates when the air from the lungs passes between them producing sound waves which are emitted from the lips and to some extent from the nose ; these are sound waves heard as speech.

(*b*) *Vocal tract.* It includes larynx, the pharnx and the nasal cavity.

Kinds of Sounds

(*i*) Voiced sound (*ii*) Unvoiced (fricative) sound.

Voiced sounds are produced by quasi-periodic pulses of air exciting the vocal tract. Unvoiced sounds are produced at some point along the vocal tract, usually towards the mouth.

There are some important speech technology areas. *viz.*, speech coding, speech enhancement, speech analysis and synthesis, speech recognition and speaker recognition.

2.2.2 Speech Technology

(*a*) **Speech coding.** "Speech Coding" is the process of capturing the speech of a person and processing it to transmit over a communication channel.

The application of "speech coding" is in the area of telephony, narrow-band cellular radio, military communication etc.

(*b*) **Speech enhancement.** This is the process of minimizing the derogatory effects of noise on the performance of speech communication, source coding etc.

The application of 'speech enhancement' is in the areas where the performance of equipment is improved in noisy atmosphere like factories etc.

(*c*) **Speech analysis and synthesis.** Analysing speech is done by studying its spectrum and extracting time-varying parameters from the signal for the production of speech.

Synthesizing speech lies in creating speech like waveforms from textual words or symbols, using a model for speech production and time-varying parameters.

The application of this are in voice alarms, reading machines for the dumb or blind, data-base enquiry services etc.

(*d*) **Speech recognition.** The process of deriving the meaning from a speech input whereby a request can be made for information or service from a machinery by conversing with it.

Application of "speech recognition" could be Banking from distant location, information retrieval systems etc.

(*e*) **Speaker recognition.** It means to recognize a particular person's identity with the sample speech dipping.

2.2.3 Parameters of Speech

(*i*) *Pitch :* Corresponds to frequency of sound (in Hz).

(*ii*) *Loudness :* This relates to intensity of sound (in dB).

(*iii*) *Quality :* This relates to harmonic constant of sound (in timbre).

'Phonemes' are the smallest unit of sound that are recognized by contrast with their environment, these are forming the basic units of speech. 'Dipones' are sounds that stretch from the middle of one phoneme to the centre of the next, there by spanning the transition region.

2.2.4 Speech Analysis

The most common methods of speech analysis are as follows :

(*a*) *Short-time fourier analysis*

(*b*) *Linear prediction.*

(*c*) *Homonorphic filtering.*

Let us discuss about these three methods of speech analysis.

(*a*) **Short-time fourier analysis.** The short-time Fourier transform of a sampled speech signal represented by the sequence $x(n)$ is given by

$$x(n) = \sum_{k=-\alpha}^{\alpha} x(k)\, h(n-k)\, e^{-j\omega k} \qquad \text{...(2.1)}$$

Fig. 2.1 shows the short-time fourier analysis.

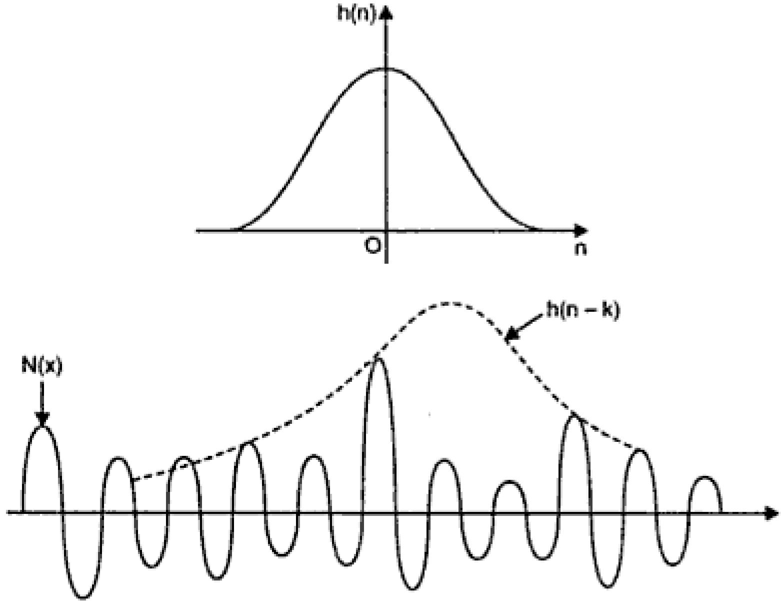


Fig. 2.1. Short-time fourier analysis.

There are two methods of obtaining the short-time fourier analysis.

(*i*) Through analog implementation of a filter bank.

(*ii*) By digitally computing short-time fourier transform either by using a filter bank or by using the FFT algorithm.

(*b*) **Linear prediction.** This method is based on "Auto regressive moving average" or pole-zero model.

If H(z) is an all-pole transfer function given by

$$H(z) = \frac{A}{1 - \sum_{k=1}^{P} a_k z^{-k}} \qquad \text{...(2.2)}$$

We have the time-series discrete time signal as

$$x(n) = \sum_{k=1}^{P} a_k\, x(n-k) + A\,\delta(n) \qquad \text{...(2.3)}$$

where A is the gain factor,

for $n > 0$,

$$x(n) = \sum_{k=1}^{P} a_k\, x(n-k) \qquad \text{...(2.4)}$$

If the data to be modelled does not correspond to the impulse response of an all-pole filter, the linear combination will give a close approximation to $x(n)$. Assume that $x(n)$ denotes this approximate value,

$$\tilde{x}(n) = \sum_{k=1}^{P} a_k\, x(n-k)\,,\ n > 0 \qquad ...(2.5)$$

The corresponding error is given by

$$e(n) = x(n) - \tilde{x}(n)$$

$$= x(n) - \sum_{k=1}^{P} a_k\, x(n-k)\,,\ n > 0 \qquad ...(2.6)$$

Mean squared error is given by

$$E_{MS} = \sum_{n=1}^{N} e^2(n) = \sum_{n=1}^{N-1}\left[x(n) - \sum_{k=1}^{P} a_k\, x(n-k)\right]^2 \qquad ...(2.7)$$

The parameters $[a_k]$ that minimize E_{MS} are determined by partial derivating E_{MS} with respect to each co-efficient a_k, $k = 1, 2, \ldots\ldots p$ and equating to zero, *i.e.*,

$$\frac{\partial E_{MS}}{\partial a_i} = 0, \quad i = 1, 2, \ldots\ldots p.$$

that gives

$$\sum_{k=1}^{p} a_k\, \phi_{ik} = \phi_{i0}\,,\ i = 1, 2, \ldots\ldots\ldots, p \qquad ...(2.8)$$

with

$$\phi_k = \sum_{k=1}^{N-1} x(n-1)\, x(n-k). \qquad ...(2.9)$$

(*c*) **Homonorphic filtering (Cepstral Analysis).** Since the excitation function and vocal tract impulse response are convolved to produce speech, this problem is thought of as a separation or deconvolution of speech into these two components.

The deconvolution speech is carried out by the non-linear filtering technique, described here as "Homonorphic filtering". The convolution operation is converted into addition which gives the output called the "complex cepstrum". Please note that the "Cepstrum" of a signal represents the fourier transform of its power spectrum.

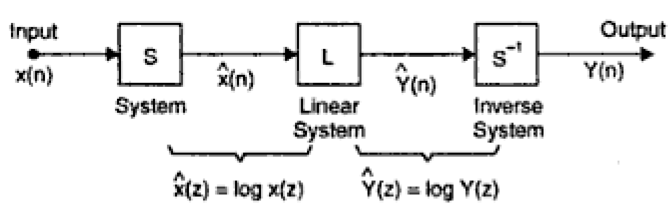


Fig. 2.2. Homonorphic systems for deconvolution.

From the Fig. 2.2, we have

$$x(n) = x_1(n) * x_2(n) \qquad ...(2.10)$$

from the system definition S.

The system S has the property

$$\hat{X}(z) = \log X(z) \qquad ...(2.11)$$

where $\hat{X}(z)$ is the z-transform of $\hat{x}(n)$.

Therefore $$X(z) = \hat{X}_1(z) + \hat{X}_2(z) \qquad ...(2.12)$$

Hence $$x(n) = \hat{x}_1(n) + \hat{x}_2(n) \qquad ...(2.13)$$

From equation 2.13, it is seen that a convolution of components is done by their addition.

Fig. 2.3 below shows a layout of homonorphic system.

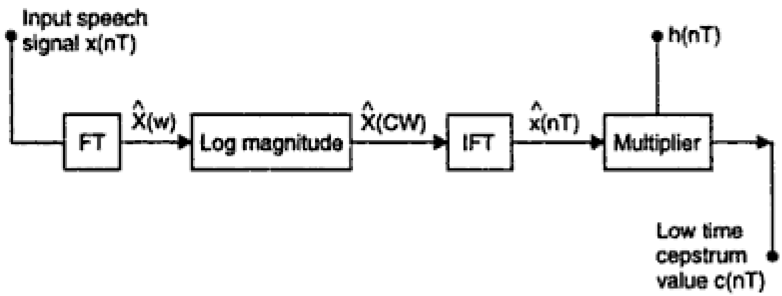


Fig. 2.3 (*a*). Analyser portion of Homonorphic system.

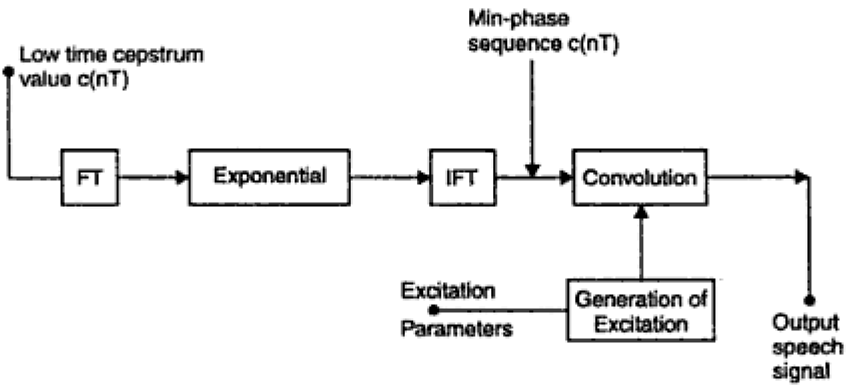


Fig. 2.3 (*b*). Synthesizer portion of Homonorphic system.

2.2.5 Speech Coding

The process of representing the speech signal in digital form at a low bit rate, with which it can be understood by a listener is called "Speech coding".

There are four important parameters of a speech coder.

(*a*) **Bit rate.** It measures the number of special properties of speech exploited.

(*b*) **Quality.** Implies the degradation of the coded speech signal.

(*c*) **Delay.** It implies the amount of speech signal needed to find the parameters of a speech coder reliably.

(*d*) **Complexity.** It measures the computational necessity for coder implementation in signal processing hardware.

There are different kinds of speech coding which is tabulated as below :

Table 2.1. Speech coding methods

Wave form coding	*Base band coding*	*Narrowband coding*
(*a*) Pulse Code Modulation (PCM) (*b*) Adaptive pulse code modulation (*c*) Linear predictive coding (*d*) Frequency domain coding	(APCM) No specific type available	(*a*) Pitch-excited coder (*b*) Segment Quantization (SQ) (*c*) Vector Quantization (VQ) (*d*) Optional Scalar Quantization (OSQ)

Waveform coding. In waveform coding, the waveform of the signal is pressured *i.e.*, the amplitude versus time waveform of the decoded signal $y(n)$ should be closely to that of the input signal $x(n)$, sample by sample.

The quantization error $e(n) = y(n) - x(n)$ ...(2.14)

The energy of the error signal $e(n)$ is minimized for any desired rate of transmission.

Transform coding. A block of input speech samples is linearly transformed through a discrete transform such as DFT or DCT ; which is computed through FFT. Bits are allocated to these transform coefficients. The coefficients are quantized and transmitted in digital form to the decoder in the receiver. Then an inverse digital transformation is performed for mapping the signal back into the time domain.

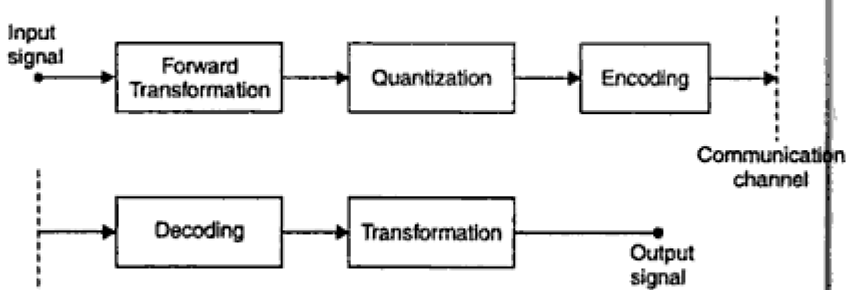


Fig. 2.4. Transform coding.

Sub-band coding. The speech signal is applied to an analysis filter bank of a set Q bandpass filters. This digital filteration divides the speech signal into a number of non-overlapping, frequency bands. These filterbands are contiguous in frequency. Therefore by

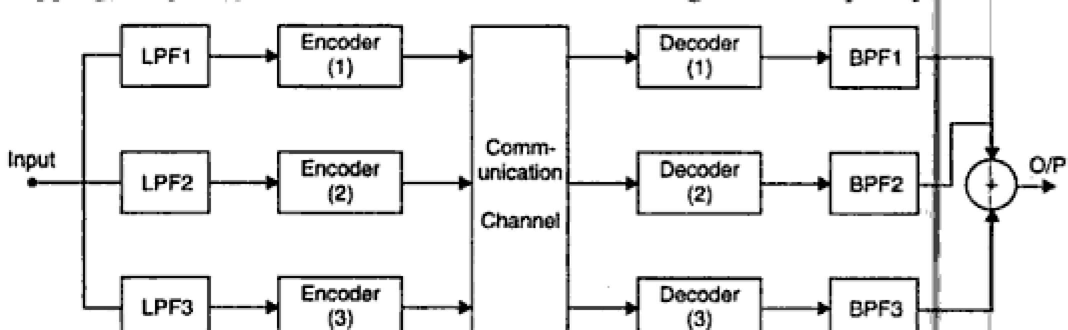


Fig. 2.5. Sub-band coding.

additive recombination of the set of sub-band signals, the original speech signal can be generated. Each band is separately quantized and coded using pulse code modulation and transmitted. The schematic is shown in Fig. 2.5.

2.3 APPLICATION TO IMAGE PROCESSING

Any function which bears two-dimensional information is called an image. Image can be represented by an array of real or complex (real and imaginary) numbers with finite number of bits with respect to speech signal (which are one-dimensional signals), image signals are two dimensional. Image can be divided into picture elements or pixels (smallest element of image). Manipulation of two-dimensional signal with the help of digital computer is called "Image Processing". Its purpose is to improve the visual appearance of Image.

A Digital Image is digitalization of picture. Normally two-dimensional image has resolution 128 × 128, 256 × 256, 512 × 512. So image can be processed using two-dimensional signal processing. The image processing including the following steps :

(*a*) Image Formation and Recording.

(*b*) Image Sampling and Quantization.

(*c*) Image Compression.

(*d*) Image Restoration.

(*e*) Image Enhancement.

All the operations are possible on advanced software artificial intelligence and high-tech digital computers. Let us discuss all the operations one by one.

2.3.1 Image Formation and Recording

The two-dimensional signal of image can be expressed by image function as

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x - x_1, y - y_1)\, f(x_1, y_1)\, dx_1\, dy_1 \qquad \text{...(2.15)}$$

Eqn. (2.15) governs a 2D linear time invariant system. Here system impulse response function $h(x - x_1, y - y_1)$ is commonly referred as point-spread function which is usually associated with optical image. The function $f(x, y)$ is the accumulation of energy from the objects radiant energy distribution.

Two major technologies are used for image sensing and recording, which are photo-chemical recording and photo-electronic recording. Both of the technologies are exemplified by readily available products which are photo-graphic films and television respectively. (Here "television" is used in generic sense not commercial broad casting television).

2.3.2 Image Sampling and Quantization

After formation and recording of an image, it is sampled and quantized for the suitability of digital processing.

In a system project, a spot of light with intensity I_1 incident on a film and intensity I_2 reflected from the film and collected by photo-multiplier. The transmittance is defined by

$$T = \frac{I_2}{I_1} \qquad \text{...(2.16)}$$

This eqn. (2.16) can be used to compute optical density. The mathematical model can also be described as a spot of light, moves in a raster to sample the film is given by

$$g_1(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_a(x - x_1, y - y_1)\, g(x_1, y_1)\, dx_1\, dy_1 \qquad ...(2.17)$$

Here h_a is the intensity profile of the spot of light projected an film. g is the image on film and finally g_1 is actual sampled image. The sample matrix $g_1(k\,\Delta x, l\,\Delta y)$ is the sampled or digital image.

2.3.3 Image Compression

In a digital image 10^5 to 10^6 data are there. The processing of these higher value of image data is a very stupendous task. But a digital image has large number of redundancy which can be reduced by image compression. So we can say that image compression is a science of efficiently coding a digital image to reduce the number of bits, which required to represent it.

Uncompressed image consumes memory space in a large amount so it increases complexity in computational and need a very large transmission bandwidth. A compressed image reduces the redundancy in image. There are mainly three types of redundancy which are discussed one by one.

The first type redundancy is Spatial Redundancy which arises due to correlation between neighbouring pixels. Second type is Spectral Redundancy which is correlation between various colour plans. And finally is Temporal Redundancy is the correlation between different frames in an image sequence.

In an Image Compression System, the original continuous time image signal is fed to A/D (Analog to Digital) converter, which converts it into digital signal. Now a serial to parallel (S/P) converter decomposed signal into parallel channels which fed to a quantizer. The S/P converter is linear transformer or filter banks are used. This quantized output is coded by the use of lossless coding device, whose output is compressed digital image signal. A simple block diagram of image compression system is shown in Fig. 2.6.

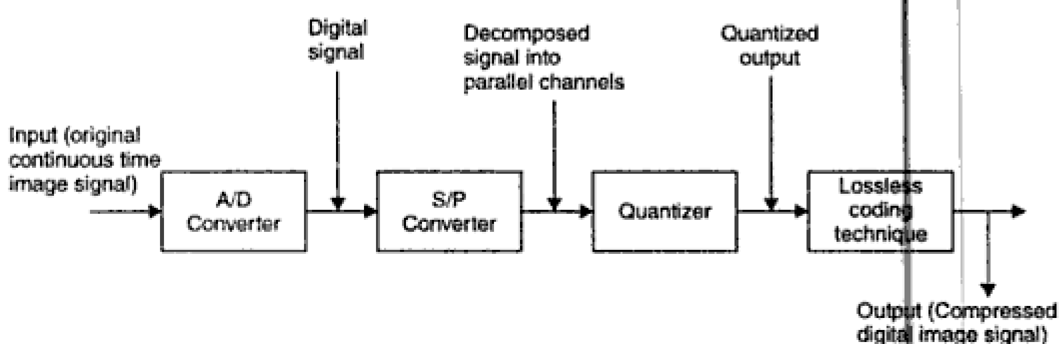


Fig. 2.6. Image compression system.

Applications of digital signal processing image compression system

There are mainly three types of compression technique based on the method of redundancy detection :

(*a*) Direct data compression method

(*b*) Transformation method

(*c*) Parametric extraction method.

2.3.4 Image Restoration

The process of image restoration is used for correcting imaging effect to recover an original signal. This type of effect (imaging effect) is due to variety of intermixing factors, which are defocusing imaging camera, relative motion between object and camera, noise in sensors etc., All types of imaging effects deteriorate image quality.

The process of image restoration is to attempt a image which should be sharp, clean and free from the degradation. The restoration process is also called Image Deblurring. The process of image formation and recording can be modelled as

$$g(x, y) = R\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x - x_1, y - y_1)\, f(x_1, y_1)\, dx_1\, dy_1\right] + n(x, y) \qquad ...(2.18)$$

Here $g(x, y)$ is the actual image, R is the response characteristic of the recording process and $n(x, y)$ is additive noise source.

In the restoration of digital image following equation can be expressed in discrete form :

$$g(p, q) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(i, j)h\,(p - i, q - j) \qquad ...(2.19)$$

A large set of simultaneous linear equations can be solved by DSP techniques such as linear filters and FFT algorithms which are computationally efficient tools for solving these.

2.3.5 Image Enhancement

This technique improves the appearance of image for human perception by choosing some image features like edges or contrast etc. Its main application is in biomedical engineering field for computer aided mammographics studies.

In image enhancement spatial filtering is mainly used whose operation is done on image to reduce noise contamination of the image signal. Image enhancement is composed of a variety of methods whose suitability depends upon the goals at hand when enhancement is originally applied.

REVIEW QUESTIONS

1. Give the areas in which signal processing find its application.
2. Explain the various stages in voice processing.
3. How is a speech signal generated ?
4. Give the model of speech production system ?
5. What is the need for short time spectral analysis ?
6. What is a vocoder ? Explain with a block diagram ?
7. Describe how targets can be detected using radar.
8. Give an expression for the following parameters related to radar
   (*a*) beam width, and
   (*b*) maximum unambiguous range.

9. Explain with the block diagram the modern radar system.
10. Give the various image processing applications.
11. Give the various coding techniques for images.
12. What is the need for image compression ?
13. Give the block diagram of basic restoration process.
14. What is sub-band coding ?
15. Explain the process of digital FM stereo signal generation.
16. Explain how privacy can be acheieved in telephone communications.

DIGITAL SIGNAL PROCESSING

Unit 2

Chapters :

3 Discrete Time Systems

3.1 DISCRETE-TIME SIGNALS AND SYSTEMS

3.1.1 Definition

1. A discrete-time signal is a sequence, that is a function defined on the positive and negative integers.
2. A discrete-time system is a mapping from the set of acceptable discrete-time signals called the input set, to a set of discrete-time signals called output set.
3. A discrete-time signal whose values are from a finite set is called a digital signal.
4. A digital system is a mapping which assigns a digital output signal to every acceptable digital input signal.

3.1.2 Representations

1. **Graphical.** In digital signal processing, signals are represented as sequence of numbers called samples. A sampled value of typical discrete-time signal or sequence is denoted by $x(n)$ which is a function of independent variable that is an integer. It is graphically represented in Fig. 3.1.

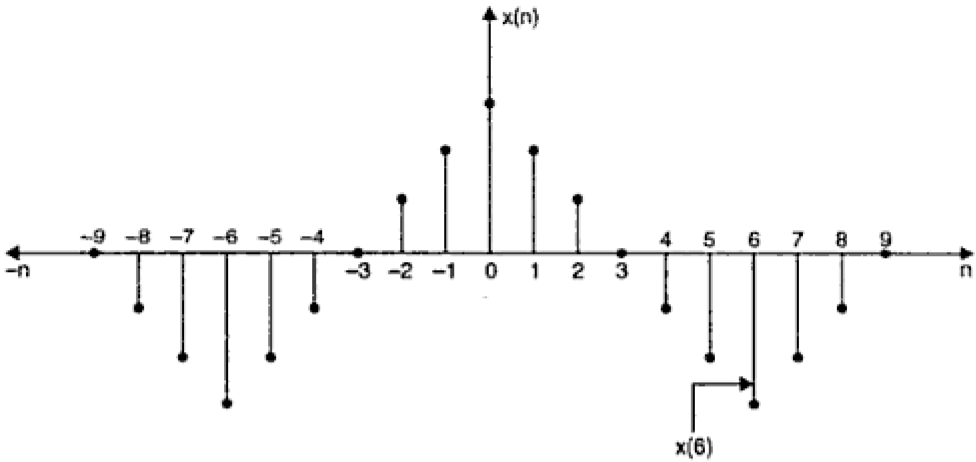


Fig. 3.1. Graphical representation.

It is important to note that $x(n)$ is defined only for integer values of n and undefined for non-integer values of n.

In the signal we have assumed that a discrete-time sequence is defined for every integer value of n for $-\infty < n < \infty$. The particular value of sequence at time k is simply denoted by $x(k)$. Here, we primarily concerned with sequence at equally spaced intervals.

2. **Alternative representation.** The discrete-time sequence may be represented in number of ways. Some of the alternative representations that are often more convenient to use. These are,

(*i*) **Functional representation** such as,

$$x(n) = \begin{cases} 2 & \text{for} \quad n = 1, 3, 5 \\ 1 & \text{for} \quad n = -1, -2, 4, 7 \\ 0 & \text{for} \quad \text{otherwise} \end{cases}$$

(*ii*) **Tabular representation**, such as

n	....	-2	-1	0	1	2	3 ......
$x(n)$		0	1	4	3	-5	-1

(*iii*) **Sequence representation.** Representation based on length the discrete time signal may be finite length or infinite length sequence. This finite length also called finite duration sequence is only defined only for a finite time interval.

The finite duration sequence can be represented,

$$x(n) = \{-2, -1, 3, \underset{\uparrow}{1}, 0, 4, 2\}$$

where, the time origin (sign origin, $n = 0$) is indicated by symbol $\uparrow$.

An finite duration sequence can be represented as

$$x(n) = \{...... 0, 0, 1, -1, 2, 4, \underset{\uparrow}{1}, 0, 0, 1\}$$

A sequence which is zero for $n < 0$ can be represented as

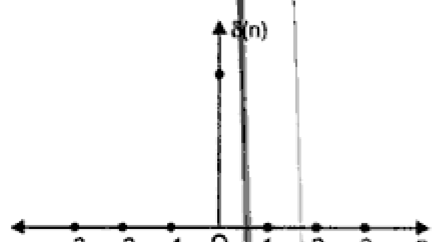
$$x(n) = \{\underset{\uparrow}{1}, 2, 0, 1, 4, 2, 0, 0\}.$$

3.1.3 Some Elementary Sequence

Any signal can be represented in terms of some basic sequences. Some of the basic sequences are defined below :

1. **Unit sample sequence (unit impulse sequence).** The unit sample sequence contains only one non-zero valued element and it is defined as,

$$\delta(n) = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

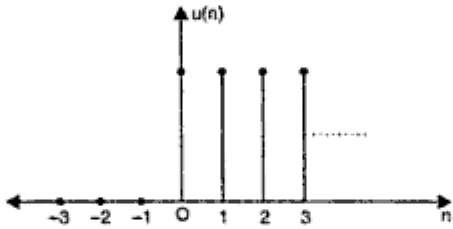


The delayed unit sampled sequence denoted by $\delta(n-k)$, has its non-zero value at sample time k, *i.e.*,

$$\delta(n-k) = \begin{cases} 1 & n = k \\ 0 & \text{otherwise} \end{cases}$$

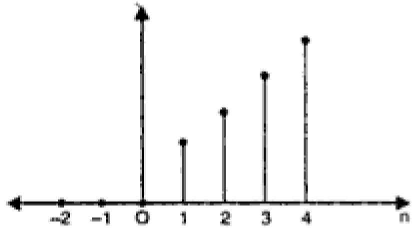
2. **Unit step sequence.** It is defined as,

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



3. **Unit ramp sequence.**

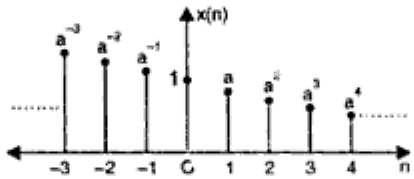
$$u_r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



(a is real)

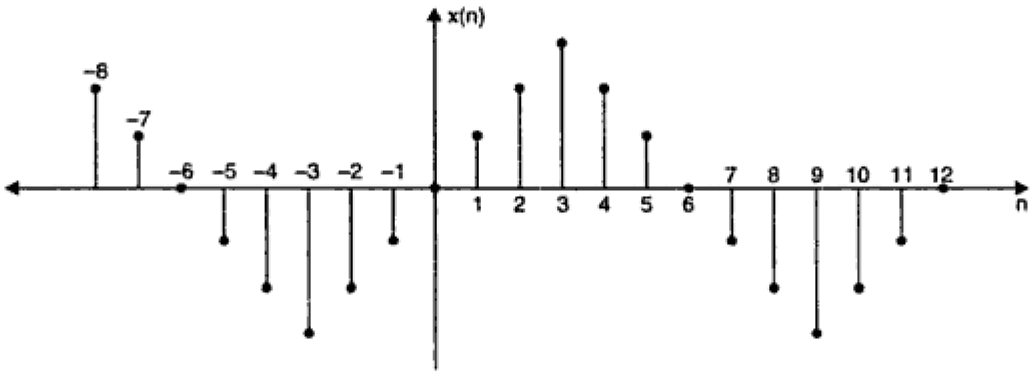
4. **Real exponential sequence.**

$x(n) = a^n$ for all 'n' $0 < a < 1$.



5. **Sinusoidal sequence.**

$x(n) = A \sin \omega_0 n.$ for all 'n'.



3.1.4 Representation of Arbitrary Sequence

An arbitrary sequence can be represented as a sum of scaled, delayed unit sample.

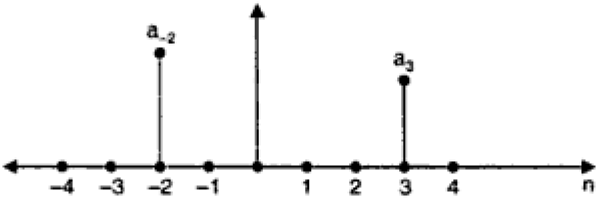


Fig. 3.2. Arbitrary sequence.

e.g. $x(n) = a_{-2}\,\delta(n+2) + a_3\,\delta(n-3)$

In general,

An arbitrary sequence can be expressed as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad ...(3.1)$$

where, $x(k)$ represents the magnitude of the k^{th} member of the sequence $x(n)$.

3.2 CLASSIFICATION OF DISCRETE-TIME SIGNAL

1. **Energy signals and power signals.** The energy signal $x(n)$ is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 .$$

We defined the signal energy of $x(n)$ over the finite interval $-N \leq n$

$$E_N = \sum_{n=-N}^{N} |x(n)|^2 \qquad ...(3.2)$$

We can express the signal energy E as,

$$E = \lim_{N\to\infty} E_N \qquad ...(3.3)$$

and the average power signal $x(n)$ as,

$$P = \lim_{N\to\infty} \frac{1}{2N+1} E_N \qquad ...(3.4)$$

If E_N is finite, $P = 0$

If E is infinite, the power P may be either finite or infinite.

2. **Periodic and aperiodic signals.** A signal $x(n)$ is periodic with period N if and only if,

$$x(n+N) = x(n) \quad \text{for all '}n\text{'}. \qquad ...(3.5)$$

The smallest value of N, for which above condition holds is called (fundamental) period. If there is no value of N that satisfies this condition is called non-periodic or aperiodic.

e.g. The sinusoidal signal of form,

$$x(n) = A\cos\omega_0 n$$
$$x(n) = A\cos 2\pi f_0 n.$$

is periodic when it is a rational no. *i.e.*, f_0 can be expressed as,

$$f_0 = \frac{k}{N}.$$

where, k and N are integer and relatively prime.

The energy of periodic signals over a single period $(0 \leq n \leq N-1)$ is finite if $x(n)$ takes on finite values over the period . However, energy of the periodic signal for $-\infty \leq n \leq \infty$ is infinite. On the other hand, the average power of periodic signal is finite.

If $x(n)$ is periodic with fundamental period N and its magnitude is finite, then its power is given by,

$$P_{av} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \qquad ...(3.6)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} |A \sin 2\pi f_0 n|^2 = \lim_{N \to \infty} \frac{A}{N} \sum_{n=0}^{N-1} (1)^2$$

$$= \lim_{N \to \infty} \frac{A}{N} . N \qquad \boxed{P_{av} = A}$$

Therefore, periodic signals are power signals.

3. **Symmetric (even) and antisymmetric (odd) signals**

A real valued signal $x(n)$ is called symmetric

if $$x(n) = x(-n),$$

on the other hand, a signal $x(n)$ is called antisymmetric

if $$x(-n) = -x(n).$$

Even signal, $$X_{cs}(n) = X_e(n) = \frac{1}{2}[X(n) + X(-n)] \qquad ...(3.7)$$

Odd signal, $$X_{ca}(n) = X_0(n) = \frac{1}{2}[X(n) - X(-n)] \qquad ...(3.8)$$

X(n) is defined as, $$\boxed{X(n) = X_e(n) + X_0(n)} \qquad ...(3.9)$$

Problem 1. *Consider a sequence defined by,*

$$x(n) = \begin{cases} 4(-1)^n & n \ge 0 \\ 0 & otherwise \end{cases}$$

determine the power and energy of x(n).

Sol. $$E = \sum_{n=0}^{\infty} |4(-1)^n|^2$$

$$= \sum_{n=0}^{\infty} |(4)^2 (1)^n| = 16 \sum_{n=0}^{\infty} 1 = 16 \left[\frac{1}{1-1}\right]$$

$$\boxed{E = \infty}$$

Power, $$P_{av} = \lim_{N \to \infty} \frac{16}{N+1} \sum_{n=0}^{\infty} (1)^n = \lim_{N \to \alpha} \frac{16(N+1)}{(N+1)}$$

$\boxed{P_{av} = 16}$ which is finite.

Problem 2. *Determine the response of the following systems to the input*

$$X(n) = \{........, 0, 3, 2, 1, 0, 1, 2, 3, 0 ....\}$$
$$\uparrow$$

(a) $y(n) = X(n + 1)$ (b) $y(n) = \frac{1}{3}[x(n + 1) + x(n) + x(n - 1)]$

(c) $y(n) = \sum_{k=-\infty}^{n} X(k)$.

Sol. (*a*) To find $y(n) = X(n+1)$

In this case the system 'advances' the input one sample into the future *i.e.*, at $n = 0$, $y(0) = x(1)$.

The response of this system to the given input is,

$$y(n) = \{\ldots\ldots\ 0, 3, 2, 1, \underset{\uparrow}{0}, 1, 2, 3, 0\ \ldots\ldots\}.$$

(*b*) To find $y(n) = 1/3\ [x(n+1) + x(n) + x(n-1)]$.

The output of the system at any time is the mean value of the present, the immediate past and the immediate future sample.

For example, the output at time $n = 0$ is,

$$y(0) = \frac{1}{3}\ [x(-1) + x(0) + x(1)]$$

$$= \frac{1}{3}\ [1 + 0 + 1]$$

$$\boxed{y(0) = 2/3}$$

Repeating this computation for every value of n, we obtain,

$$y(n) = \{\ \ldots\ldots\ 0, 1, 5/3, 2, 1, \underset{\uparrow}{2/3}, 1, 2, 5/3, 1, 0 \ldots\ldots\}.$$

(*c*) $y(n) = \sum_{k=-\infty}^{n} x(k)$

This system is basically an accumulator that computes the running sum of all the past input values upto present time. The response of the system to the given input is,

$$y(n) = \{\ldots\ldots\ 0, 3, 5, 6, 6, 7, 9, 12, 12, \ldots\ldots\}.$$

3.3 SAMPLING

In some applications a discrete-time sequence $x(n)$ is generated by periodically sampling a continuous time signal $x_a(t)$ at uniform time intervals.

$$x(n) = x_a(t)\big|_{t=nT} = x_a(nT)\ ;\ n = \ldots\ldots -2, -1, 0, 1, 2 \ldots\ldots \qquad \ldots(3.10)$$

The spacing T between two consecutive samples in eqn. (3.10) is called the sampling interval or sampling period.

The reciprocal of the sampling interval T, denoted as F_T is called the sampling frequency.

$$F_T = \frac{1}{T}.\ \text{(cycle/sec or Hertz)}.$$

3.4 REAL AND COMPLEX SEQUENCE

Real sequence. If $X(n)$ is real for all values of n, then $\{x(n)\}$ is a real sequence.

Complex sequence. If the n^{th} sample value is complex for one or more values of n, then it is complex sequence. It can be defined as,

$$\{x(n)\} = \{x_{re}(n)\} + j\{x_{im}(n)\}. \quad ...(3.11)$$

where, $x_{re}(n)$ and $x_{im}(n)$ are the real part and the imaginary part of $x(n)$.

Complex conjugate sequence

$$\{x^*(n)\} = \{x_{re}(n)\} - j\{x_{im}(n)\}. \quad ...(3.12)$$

3.5 FINITE AND INFINITE LENGTH SEQUENCE

The discrete-time signal may be a finite-length or an infinite-length sequence. A finite length sequence is defined only for a finite time interval.

$$N_1 \le n \le N_2 \quad ...(3.13)$$

where, $-\infty < N_1$ and $N_2 < \infty$ with $N_2 \ge N_1$.

The length of duration N of the above finite-length sequence is,

$$N = N_2 - N_1 + 1 \quad ...(3.14)$$

Zero-padding. The process of lengthening a sequence by adding zero-valued samples is called appending with zeros or zero-padding.

3.6 TYPES OF INFINITE-LENGTH SEQUENCE

1. **Right-sided sequence.** A right-sided sequence $x[n]$ has zero-valued samples for $n < N_1$ *i.e.*,

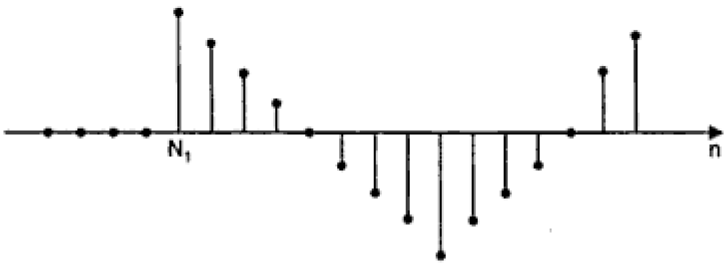


Fig. 3.3. Right-sided sequence.

$$x(n) = 0 \text{ for } n < N_1 \quad ...(3.15)$$

where, N_1 is a finite integer that can be positive or negative.

2. **Anti causal sequence and causal sequence.** If $N_1 \ge 0$, a right-sided sequence is usually called a causal sequence.

If $N_2 \le 0$, a left-sided sequence is usually called a anticausal sequence.

3. **Left-sided sequence.** A left-sided sequence $x[n]$ has zero-valued samples for $n > N_2$ *i.e.*,

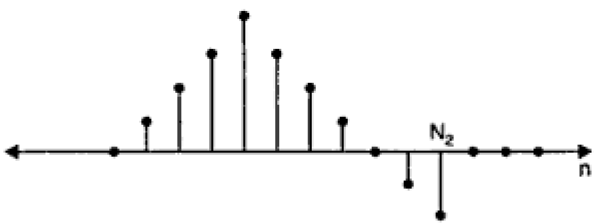


Fig. 3.4. Left-sided sequence.

$$x(n) = 0 \text{ for } n > N_2$$

where, N_2 is a finite integer which can be positive or negative.

In general

Two sided-sequence is defined for all values of n in the range $-\infty < n < \infty$.

3.7 OPERATIONS ON SEQUENCES

A single-input, single output discrete time system operates on a sequence, called the input sequence, according to some prescribed rules and develops another sequence, called the output sequence.

Basic operations

1. **Modulation or Multiplication.** Let $x(n)$ and $y(n)$ be two known sequence. By forming the product of the sample values of these two sequence at each instant, we form a new sequence $w_1(n)$.

$$\boxed{w_1(n) = x(n) \cdot y(n)} \qquad \ldots(3.16)$$

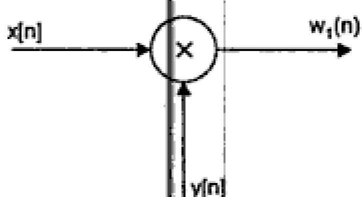


Fig. 3.5. Modulation.

In some applications, the product operation is also known as modulation. The device implementing the modulation operation is called a 'modulator', and its schematic representation is shown in Fig. 3.5.

(*i*) *Windowing.* An application of the product operation is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called a window sequence. This process of forming the finite length sequence is usually called windowing. It is basically used in design of digital filter.

(*ii*) *Addition.* The two sequences $x_1(n)$ and $x_2(n)$ can be added or subtracted, resulting in a new sequence,

$$\boxed{w_2(n) = x(n) + y(n)} \qquad \ldots(3.17)$$

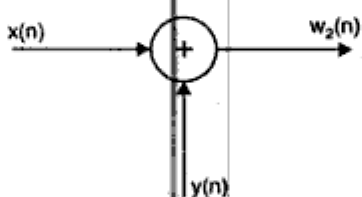


Fig. 3.6. Addition of two sequence.

The device implementing the addition operation is called an adder.

(*iii*) *Scalar multiplication.* A new sequence is generated by multiplying each sample of a sequence $x(n)$ by a scalar A.

$$\boxed{w_3(n) = Ax(n)} \qquad \ldots(3.18)$$

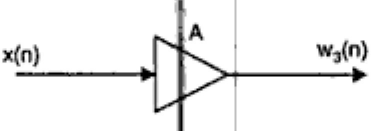


Fig. 3.7. Scalar multiplication.

The device implementing the multiplication operation is called a multiplier.

(*iv*) *Time shifting operation.* The shift operation takes the input sequence and shift the values by an integer increment of the independent variable.

$$w_4(n) = x(n - N) \qquad \ldots(3.19)$$

where, N is an integer.

The shifting may delays or advances the sequence in time. If $N > 0$, it is a delaying operation. If $N < 0$, it is an advancing operation.

The device implementing the delay operation by one sample is called a "**Unit delay**" and its schematic representation is shown in Fig. 3.8

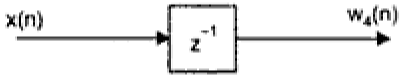


Fig. 3.8. Unit delay.

$$\boxed{w_4(n) = x[n-1]}$$

The schematic representation of the unit advance operation is shown in Fig. 3.9

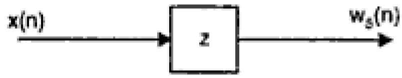


Fig. 3.9. Advance operation.

$$\boxed{w_5[n] = x[n+1]}$$

(*v*) *Time-reversed or Folding.* The time reversal operation, also called the folding operation, is another useful scheme to develop a new sequence.

$$\boxed{w_6(n) = x(-n)} \qquad \ldots(3.20)$$

which is the time-reversed version of the sequence $x(n)$.

(*vi*) *Pick-off node.* It is used to provide multiple copies of a sequence.

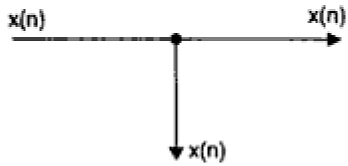


Fig. 3.10. Pick-off node.

Problem 3. *Consider the following two sequence of length 5 defined for $0 \le n \le 4$:*

$$c(n) = \{3.2, 41, 36, -9.5, 0\}$$
$$d(n) = \{1.7, -0.5, 0, 0.8, 1\}.$$

Determine $w_1(n)$, $w_2(n)$ and $w_3(n) = \frac{7}{2} c(n)$.

Sol. (1) $w_1(n) = c(n) \,.\, d(n)$.

$w_1(n) = \{5.44, -20.5, 0, -7.6, 0\}$.

(2) $w_2(n) = c(n) + d(n)$

$= \{4.9, 40.5, 36, -8.7, 1\}$

(3) $w_3(n) = \frac{7}{2} c(n)$

$w_3(n) = \{11.2, 143.5, 126, -33.25, 0\}$.

Problem 4. *Consider a sequence $[g(n)]$ of length 3 defined for $0 \le n \le 2$ given by, $[g(n)] = \{-21, 1.5, 3\}$, and a sequence of length 5 defined for $c(n) = \{3.2, 41, 36, -9.5, 0\}$ Find $c(n) \,.\, g(n)$ and $c(n) + g(n)$.*

Sol. We can develop another sequence $g(n)$ by operating on this sequence of length 5 and defined for $0 \le n \le 4$ by appending it with two zero-valued samples

$$g_e(n) = \{-21, 1.5, 3, 0, 0\}.$$

we can generate,

(1) $c(n) \,.\, g_e(n) = \{-67.2, 61.5, 108, 0, 0\}$

(2) $c(n) + g_e(n) = \{-17.8, 42.5, 39, -9.5, 0\}$.

Problem 5. *Find the $y(n)$ from the following sequence is given by,*

(*i*) *$x(n) = \{1, 2, 1, -1\}$ and $a = 2$.*

(*ii*) *$x_1(n) = \{-1, 2, -3, -2\}$ and $x_2(n) = \{1, -1, -2, 1\}$*

find $x_1(n) \,.\, x_2(n) = ?$

Sol. (*i*) $y(n) = ax(n)$

$y(n) = \{2, 4, 2, -2\}$

(*ii*) $y(n) = x_1(n) \cdot x_2(n)$

$= \{-1, -2, 6, -2\}$.

3.8 SAMPLING RATE ALTERATION

Another useful operation is the sampling rate alteration that is employed to generate a new sequence with sampling rate higher or lower than that of a given sequence.

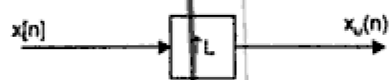
Thus if $x[n]$ is a sequence with a sampling rate of F_T Hz and it is used to generate another sequence $y(n)$ with a desired sampling rate of F'_T Hz, then the sampling rate alteration ratio is given by,

$$\frac{F'_T}{F_T} = R \qquad \ldots(3.21)$$

- If $R > 1$, the process is called "interpolation" and results in a sequence with a higher sampling rate.
- On the other hand, if $R < 1$, the sampling rate is decreased by a process called "decimation".

The basic operations employed in the sampling rate alteration process are called up-sampling and down-sampling. These operations play important roles in multirate discrete time systems.

Up-sampling. In up-sampling by an integer factor $L > 1$, $L - 1$ equidistant zero-valued samples are inserted by the up-sampler between each two consequtive samples of the input sequence $x[n]$ to develop an output sequence $y(n)$ according to the relation,



$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L\ldots\ldots \\ 0 & \text{otherwise} \end{cases} \qquad \ldots(3.22)$$

Sampling rate of $y(n)$ is 'L' times larger than that of the original sequence $x(n)$.

Down-sampling. Conversely, the down sampling operation by an integer factor $M > 1$, on a sequence $x[n]$ consists of keeping every M^{th} sample of $x[n]$ and removing $M - 1$ in between samples, generating an output sequence $y(n)$ according to the relation,

$$y(n) = x[nM] \qquad \ldots(3.23)$$

The result in a sequence $y(n)$ whose sampling rate is $(1/m)^{th}$ that of $x(n)$.

3.9 CLASSIFICATION BASED ON SYMMETRY PROBLEM

Problem 6. *Consider the finite length sequence of length 7 defined for* $-3 \le n \le 3$:

$$\{g(n)\} = \{0,\ 1 + j4,\ -2 + j3,\ \underset{\uparrow}{4 - j2},\ -5 - j6,\ -j2,\ 3\}$$

To determine its g_{cs} *and* g_{ca}.

Sol. (1) To determine conjugate symmetric part $g_{cs}(n)$

$$[g^*(n)] = \{0,\ 1 - j4,\ -2 - j3,\ \underset{\uparrow}{4 + j2},\ -5 + j6,\ j2,\ 3\}$$

Whose time-reversed version is given by,

$$\{g^*(-n)\} = \{3, j2, -5+j6, \underset{\uparrow}{4+j2}, -2-j3, 1-j4, 0\}$$

$$g_{cs}(n) = \frac{1}{2}[x(n) + x^*(-n)].$$

$$\{g_{cs}(n)\} = \{1.5, 0.5+j3, -3.5+j4.5, \underset{\uparrow}{4}, -3.5-j4.5, 0.5-j3, 1.5\}$$

2. To determine conjugate anti-symmetric part $g_{ca}(n)$

$$\{g_{ca}(n)\} = \frac{1}{2}[x(n) - x^*(-n)].$$

$$\{g_{ca}(n)\} = \{-1.5, 0.5+j1, 1.5-j1.5, \underset{\uparrow}{-j2}, -1.5-j1.5, -0.5-j1, 1.5\}$$

It can be easily verified that,

$$g_{cs}(n) = g_{cs}^*(-n) \quad \text{and}$$
$$g_{ca}(n) = -g_{ca}^*(-n).$$

3.9.1 Periodic Conjugate-Symmetric Part and Periodic Conjugate Anti-symmetric Part

Periodic conjugate-symmetric part defined by,

$$x_{pcs}(n) = \frac{1}{2}[x(n) + x^*[(-n)_N]]. \quad , 0 \le n \le N-1 \qquad \text{...(3.24)}$$

Periodic conjugate anti-symmetric part is defined by,

$$x_{pca}(n) = \frac{1}{2}[x(n) - x^*[(-n)_N]], \quad 0 \le n \le N-1 \qquad \text{...(3.25)}$$

So that $\quad x(n) = x_{pcs}(n) + x_{pca}(n), \quad 0 \le n \le N-1 \qquad$ ...(3.26)

Note. A length N sequence $x[n]$ defined for $0 \le n \le N-1$ is said to be periodic conjugate symmetric if $x(n) = x^*[(-n)_N] = x^*[N-n]$, and is said to be periodic conjugate-antisymmetric if $x(n) = -x^*[(-n)_N] = -x^*[N-n]$.

Problem 7. *Consider finite-length sequence of length 4 defined for $0 \le n \le 3$:*

$$\{u(n)\} = \{1+j4, -2+j3, 4-j2, -5-j6\}.$$

To determine its periodic symmetric part $u_{pcs}(n)$ and its periodic conjugate antisymmetric part $u_{pca}(n)$.

Sol. To find $u_{pcs}(n)$:

$$u_{pcs}(n) = \frac{1}{2}[u(n) + u^*[(-n)_N]]$$

$$\{u^*(n)\} = \{1-j4, -2-j3, 4+j2, -5+j6\}$$

To compute modulo-4 time-reversed version $\{u^*(<-4>)\}$.

We observe that,

$$u^*[(-0)_4] = u^*(0) = 1-j4.$$
$$u^*[(-1)_4] = u^*(3) = -5+j6.$$
$$u^*[(-2)_4] = u^*(2) = 4+j2$$
$$u^*[(-3)_4] = u^*(1) = -2-j3.$$

$$\{u^*[(-n)_4]\} = \{1 - j4, -5 + j6, 4 + j2, -2 - j3\}$$

$$u_{pcs}(n) = \frac{1}{2}[u(n) + u^*(-n)_4]$$

$$u_{pcs}(n) = \{1, -3.5 + j\,4.5\,, 4\,, -3.5 - j\,4.5\}$$

$$u_{pca}(n) = \{j4\,, 1.5 - j1.5, -j2, -1.5 - j1.5\}.$$

It can be easily verified that,

$$u_{pcs}(n) = u^*_{pcs}[(-n)_4] \quad \text{and}$$

$$u_{pca}(n) = -u^*_{pca}[(-n)_4].$$

3.10 SAMPLING PROCESS

The discrete-time sequence is developed by uniformly sampling a continuous time signal $x_a(t)$ as illustrated in Fig. (3.11).

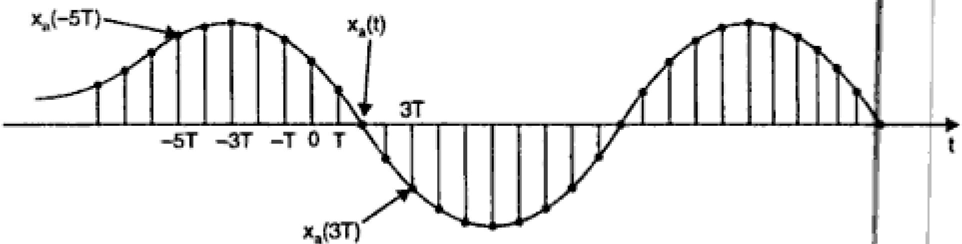


Fig. 3.11. Continuous time signal.

The relation between the two signals is given by eqn.

$$x(n) = x_a(t)|_{t=nT} = x_a(nT)\,,\ n = 0, -2, -1, 0, 1\,, \ldots\ldots \qquad \ldots(3.27)$$

where, t-time variable of the continuous time signal is related to the time variable n of the discrete time signal only at discrete-time instants t_n given by,

$$t_n = nT$$

$$= \frac{n}{F_T} = \frac{2\pi n}{2\pi F_T} = \frac{2\pi n}{\Omega_T} \qquad \ldots(3.28)$$

with, $F_T = \frac{1}{T}$ denoting the sampling frequency

and $\Omega_T = 2\pi F_T$ denoting the sampling angular frequency

For example, if continuous – time signal is,

$$x_a(t) = A\cos(2\pi f_0 t + \phi)$$

$$x_a(t) = A\cos(\Omega_0 t + \phi)$$

the corresponding discrete-time signal is given by,

$$x[n] = x_a[nT]$$

$$= A\cos[\Omega_0 nT + \phi]$$

$$= A\cos\left[\frac{2\pi\Omega_0 n}{\Omega_T} + \phi\right]$$

$$\boxed{x(n) = A\cos[\omega_{0n} + \phi]} \qquad \ldots(3.29)$$

where,

$$\omega_0 = \frac{2\pi\Omega_0}{\Omega_T} = \Omega_0 T \qquad ...(3.30)$$

It is the normalised angular frequency of the discrete-time signal $x(n)$.

Units

The unit of the normalised **digital** angular frequency ω_0 is radians per sample.

While, the unit of the normalised **analog** angular frequency Ω_0 is radians per sample and the unit analog frequency f_0 is hertz is the unit of the sampling period T is in seconds.

Problem 8. *Consider the three sequence generated by uniformly sampling the three co-sine functions of frequencies 3Hz, 7Hz and 13Hz respectively : $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$ with sampling rate of 10 Hz. i.e., with $T = 0.1$ sec. Find the derived sequence or discrete sequence.*

Sol.

$$g(t) = \cos(\Omega_0 t);$$

$$g(n) = \cos(\Omega_0 nT)$$

$$g_1(n) = \cos(6\pi n \times T)$$

$$g_1(n) = \cos(0.6\pi n)$$

Similarly,

$$g_2(n) = \cos(1.4\pi n)$$

and

$$g_3(n) = \cos(2.6\pi n).$$

Problem 9. *Determine the discrete time signal $v(n)$ obtained by uniformly sampling at sampling rate of 200 Hz., a continuous time signal $v_a(t)$ composed of a weighted sum of five sinusoidal of frequencies 30 Hz, 150 Hz, 170 Hz, 250 Hz and 330 Hz as given below :*

$$v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t).$$

Sol. To find the sampling period (T) :

$$T = \frac{1}{F} = \frac{1}{200} = 0.005 \text{ sec.}$$

The generated discrete-time signal $v(n)$ is given by,

$$v(n) = 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n) + 4\cos(2.5\pi n) + 10\sin(3.3\pi n).$$

$$= 6\cos(0.3\pi n) + 3\sin[(2\pi - 0.5\pi)n] + 2\cos[(2\pi - 0.3\pi)n] + 4\cos[(2\pi + 0.5\pi)n] + 10\sin[(4\pi - 0.7\pi)n]$$

$$= 6\cos[0.3\pi n] - 3\sin[0.5\pi n] + 2\cos[0.3\pi n] + 4\cos[0.5\pi n] - 10\sin[0.7\pi n]$$

$$v(n) = [8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)]$$

The discrete-time signal $v(n)$ is composed of a weighted sum of three-discrete-time sinusoidal signals of normalised angular frequencies : 0.3π, 0.5π and 0.7π].

3.11 CLASSIFICATION OF DISCRETE-TIME SYSTEMS

Discrete-time systems are classified according to their general properties and characteristics.

They are

(1) Static and Dynamic systems.

(2) Time-variant and time-invariant systems.

(3) Causal and non-causal systems.

(4) Stable and unstable systems.

(5) Linear and non-linear systems.

(6) FIR and IIR systems.

1. Static and Dynamic Systems

A discrete-time system is called static or "memory less" if its output at any instants 'n' depends on the input samples at the same time, but not an past or future samples of the input.

In any other case, the system is said to be dynamic or to have memory.

The systems described by the following equations,

$$y(n) = ax(n)$$
$$y(n) = ax^2(n).$$

are both static as they won't require memory. On the other hand, the systems described by the following equations

$$y(n) = x(n-1) + x(n-2)$$
$$y(n) = x(n) + x(n-1)$$

are dynamic systems as they require finite memory.

2. Time-variant and Time-invariant Systems

A system is called time-invariant if its input-output characteristics do not change with time. A "linear time-invariant" (LTI) discrete-time system satisfies both the linearity and the time-invariance properties.

To test if any given system is time-invariant, first apply an arbitrary sequence $x(n)$ and find $y(n)$.

$$y(n) = T[x(n)].$$

Now delay the input sequence by k samples and find output sequence, denote it as,

$$y(n, k) = T\,[x\,(n-k)] \quad \ldots(3.31)$$

Delay the output sequence by k samples, denote it as $y(n-k)$. If

$$y(n, k) = y(n-k) \quad \ldots(3.32)$$

For all possible values of k, the system is time-invariant on the other hand the output,

$$y(n, k) \neq y(n-k) \quad \ldots(3.33)$$

Even for one value of k, the system is time-variant

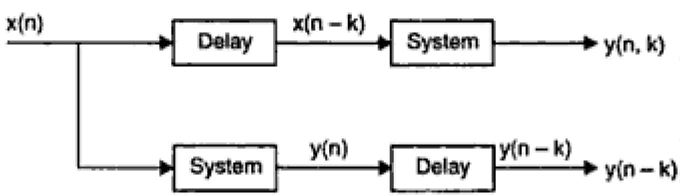


Fig. 3.12. Time invariant and time variant system.

Problem 10. *Determine if the following systems are time-invariant or time-variant.*

(i) $y(n) = x(n) + x(n-1)$ (ii) $y(n) = x(-n)$

(iii) $y(n) = x(2n)$ (iv) $y(n) = x(n) \sin \omega_0 n$.

Sol. (*i*) $\mathbf{y(n) = x(n) + x(n-1)}$.

We know that, $y(n) = T[x(n)]. = x(n) + x(n-1)$.

If the input is delayed by k units in time, we have,

$$y(n, k) = T[x(n-k)]$$

$$y(n, k) = x(n-k) + x(n-k-1) \quad ...(1)$$

If we delay the output by k units in time then,

$$y(n-k) = x(n-k) + x(n-k-1) \quad ...(2)$$

(1) = (2)

Here, $y(n, k) = y(n-k)$

So, the system is time-invariant.

(*ii*) $\mathbf{y(n) = x(-n)}$

If the input is delayed by k units in time and applied to the system, we have,

$$y(n, k) = T[x(n-k)] = x[-n-k] \quad ...(3)$$

If the output is delayed by k samples,

$$y(n-k) = x[-(n-k)] = x[-n+k] \quad ...(4)$$

(3) ≠ (4)

Here, $y(n, k) \neq y(n-k)$

So, the system is time-variant.

(*iii*) $\mathbf{y(n) = x(2n)}$

The system is described by input output equation,

$$y(n) = T[x(n)]$$

$$y(n) = x(2n)$$

If the input is delayed by k unit in time and applied to the system,

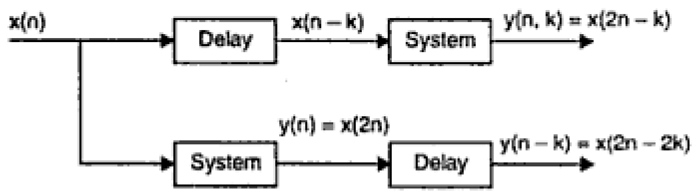
$$y(n, k) = T[x(n-k)]$$

$$y(n, k) = x(2n-k) \quad ...(1)$$

Now, if we delay the output $y(n)$ by k unit in time, the result will be

$$y(n-k) = x[2(n-k)] = x[2n-2k] \quad ...(2)$$

Since $y(n, k) \neq y(n-k)$, the system is time variant.



(*iv*) $\mathbf{y(n) = x(n) \sin \omega_0 n}$

If the *i/p* is delayed by k unit in time and applied to the system,

$$y(n, k) = x(n-k) \sin \omega_0 n.$$

If we delay the output by k unit in time, then $y(n-k) = x(n-k) \sin \omega_0 (n-k)$

Since $y(n-k) \neq y(n, k)$. So the system is time variant.

3. Causal and Non-causal System

In a discrete-time system the n_0th output sample $y[n_0]$ depends only on input samples $x[n]$ for $n \leq n_0$ and does not depend on input samples for $n > n_0$. This means that, a system is said to be causal if the output of the system at any time n depends only on present and past inputs, but does not depend on future inputs. This can be expressed mathematically as,

$$y(n) = F[x(n), x(n-1), x(n-2) ......].$$

If a system depends not only on present and past inputs but also on future inputs, then it is said to be a non-causal system.

4. Stable System

There are various definitions of stability we define a discrete-time system to be stable if and only if, for every bounded input, the output is also bounded. This implies that, if the response to $x(n)$ is the sequence $y(n)$ and if,

$$[x(n)] < B_x \quad \text{or} \quad |x(n)| < M_x \qquad ...(3.34)$$

for all values of n, then

$$[y(n)] < B_y \quad \text{or} \quad |y(n)| < M_y \qquad ...(3.35)$$

for all values of n

where B_x and B_y are finite constants. This type of stability is usually referred to as bounded-input, bounded-output (BIBO) stability.

Problem 11. *Determine whether the following systems are*

(i) Stable *(ii) Causal.*

(a) $y(n) = e^{x(n)}$ *(b)* $y(n) = nx(n)$.

Sol. (*a*) $\mathbf{y(n) = e^{x(n)}}$

(*i*) If $x(n)$ is bounded, say $|x(n)| < M$, then $|y(n)| = e^{|M|} < \alpha$.

That is, this system produces bounded output for bounded input. So it is BIBO stable system.

(*ii*) Since the output of this system depends only on the present input $x(n)$, it is a causal system.

(*b*) $\mathbf{y(n) = nx(n)}$

(*i*) To show that the system is not BIBO stable requires the specification of bounded input that produces an unbounded output.

For example, $x(n) = u(n)$, the unit step function produces an output $y(n) = nu(n)$ that grows without bounded therefore the system is not BIBO stable.

(*ii*) To determine causality we can compute $y(n)$ at an arbitrary n, say n_0.

$$y(n) = n_0\, x(n_0)$$

The output at time n_0, depends upon the input at time n_0, and n_0 future input values, therefore, the system is causal.

Problem 12. *Determine if the system described by the following equations are causal or non-causal.*

(i) $y(n) = x(n) + \dfrac{1}{x(n-1)}$ *(ii)* $y(n) = x(n^2)$.

Sol. (*i*) $\mathbf{y(n) = x(n) + \dfrac{1}{x(n-1)}}$

For $n = -1$, $y(-1) = x(-1) + \frac{1}{x(-2)}$

For $n = 0$, $y(0) = x(0) + \frac{1}{x(-1)}$

For $n = 1$, $y(1) = x(1) + \frac{1}{x(0)}$.

For all the values of n the output depends on present and past inputs. Therefore, the system is causal.

(*ii*) **$y(n) = x(n^2)$**

For $n = -1, y(-1) = x(1)$

For $n = 0$, $y(0) = x(0)$

For $n = 1$, $y(1) = x(1)$.

For negative values of n, the system depends on future inputs. So, the system is non-causal.

5. Linear and Non-linear System

A system that satisfies the superposition principle is said to be a linear system. Super position principle states that, the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of the outputs of the system to each of the individual input signals.

A system is linear if and only if

$$T[a_1x_1(n) + a_2x_2(n)] = a_1\, T[x_1(n)] + a_2\, T[x_2(n)] = a_1y_1(n) + a_2y_2(n) \quad ...(3.36)$$

for any arbitrary constants a_1 and a_2.

A relaxed system that does not satisfy the superposition principle is called non-linear.

Problem 13. *Determine whether the systems described by the following input-output equations are*

(i) linear *(ii) time-invariant*

(a) $y(n) = nx(n)$ *(b)* $y(n) = ax(n) + b$.

Sol. (*a*) **$y(n) = nx(n)$**

(*i*) We have to take two input sequence $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$$y_1(n) = T[x_1(n)] = nx_1(n)$$
$$y_2(n) = T[x_2(n)] = nx_2(n).$$

A linear combination of the two *i/p* sequence result in the output,

$$T[x_1(n) + x_2(n)] = n[x_1(n) + x_2(n)] \quad ...(1)$$

On the other hand, a linear combination of the two outputs results in the output,

$$y_1(n) + y_2(n) = nx_1(n) + nx_2(n)$$
$$y_1(n) + y_2(n) = n[x_1(n) + x_2(n)] \quad ...(2)$$

(1) = (2)

∴ The system is linear.

(*ii*) $y(n) = T[x(n)] = nx(n)$

The response of this system to $x(n - n_0)$ is,

$$T[x\,(n - n_0)] = nx(n - n_0)\,.$$

Now if we delay $y(n)$ by n_0 units in time,

$$y(n - n_0) = (n - n_0)\, x(n - n_0)$$

Since, $y(n-n_0) \neq T[x(n-n_0)]$ the system is time invariant

(*b*) **y(n) = ax(n) + b**

(*i*) For $x_1(n)$ and $x_2(n)$, the outputs are,

$$y_1(n) = ax_1(n) + b$$

$$y_2(n) = ax_2(n) + b$$

$\therefore$ $T[x_1(n) + x_2(n)] = a[x_1(n) + x_2(n)] + b$ ...(1)

But, $y_1(n) + y_2(n) = a\,[x_1(n) + x_2(n)] + 2b$ ...(2)

(1) $\neq$ (2)

$\therefore$ The system is non linear.

(*ii*) $y(n) = ax(n) + b.$ ...(3)

The response to delayed input $x(n-n_0)$ is,

$$T[x(n-n_0)] = ax\,(n-n_0) + b \quad ...(4)$$

and the delayed version of eqn. (3) is,

$$y(n-n_0) = ax(n-n_0) + b \quad ...(5)$$

Since (4) = (5), the system is time-invariant .

3.12 TIME-DOMAIN CHARACTERIZATION

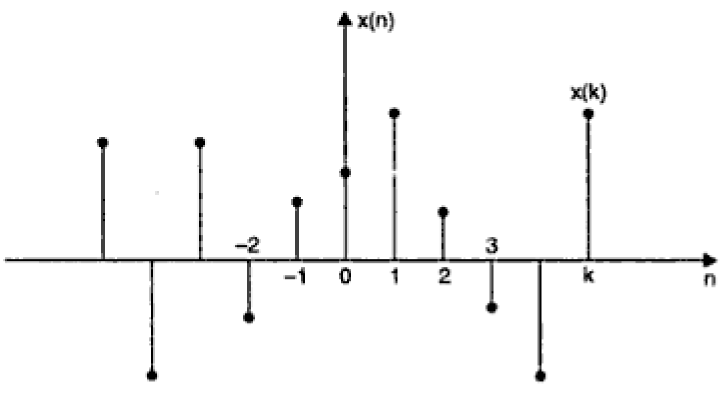
We shall discuss that LTI systems are characterized in the time domain simply by their response to a unit sample sequence. We shall also discuss that an arbitrary input signal can be decomposed and represented as a weighted sum of unit sample response.

The general form of the expression that relates the unit sample response of the system and the arbitrary input signal to the output signal, called the "convolution sum", is also derived. Thus we are able to determine the output of any linear time invariant system to any arbitrary input signal.

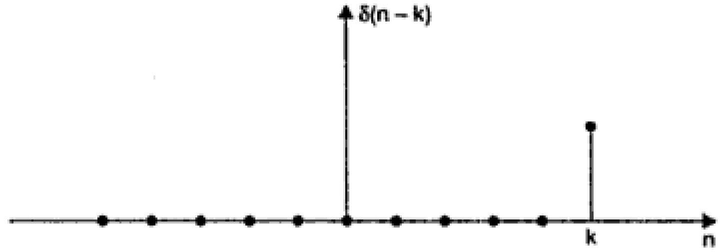
3.12.1 Representation of a Discrete-time Signal in Terms of Impulse

A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely specified by its impulse-response. *i.e.*, knowing the impulse response, we can compute output of the system to any arbitrary input.

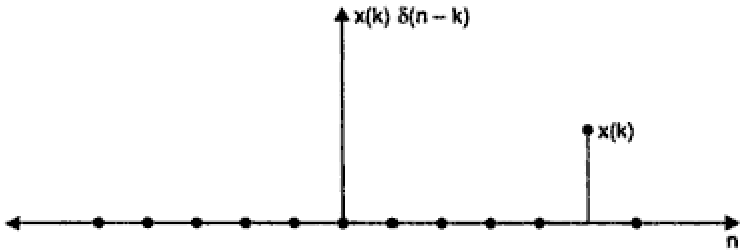
Consider a signal $x(n)$ shown in Fig. (3.13) that we wish to resolve into the sum of unit sample sequences.



(*a*)



(b)



(c)

Fig. 3.13. Multiplication of a signal $x(n)$ with a shifted unit sample sequence.

We select the elementary signal $x_k(n)$ to be,

$$x_k(n) = \delta(n - k) \qquad ...(3.37)$$

where k represents the delay of the unit sample sequence.

Now suppose that we multiplied the sequences $x(n)$ and $\delta(n-k)$. We know that $\delta(n-k)$ is zero except at $n = k$, where its value is unity. The multiplication of these two sequences results in another sequence which is zero every where expect $n = k$, where its value is $x(k)$, which is shown in above Fig. (3.13).

Thus,
$$x(n)\,\delta(n-k) = x(k)\,\delta(n-k) \qquad ...(3.38)$$

represents a sequence which is zero except at $n = k$, where its value is $x(k)$.

Similarly, if we repeat for any other delay $l(k \neq l)$ then the multiplication of $x(n)$ with $\delta(n - l)$ will result in a new sequence that is zero everywhere expect at $n = l$, Thus,

$$x(n)\,\delta(n-l) = x(l)\,\delta(n-l).$$

Therefore if we repeat this multiplication over all possible delays and sum all the resultant product sequences then the result will be sequence equal to the sequence $x(n)$. *i.e*,

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)} \qquad ...(3.39)$$

Problem 14. *A finite duration sequence x(n) is given by,*

$$x(n) = \{0.5,\ 1.5,\ \underset{\uparrow}{0},\ -1,\ 1,\ 0.75,\ 2\}$$

resolve the above sequence into a sum of weighted impulse sequence.

Sol. The above sequence is non-zero for the

$$n = \{-2, -1, 1, 2, 3, 4\}$$

$$x(n) = 0.5\,\delta(n+2) + 1.5\,\delta(n+1) - \delta(n-1) + \delta(n-2) + 0.75\,\delta(n-3) + 2\,\delta(n-4).$$

3.12.2 Discrete-time Unit Impulse Response and Convolution Sum Representation of LTI System

In sec. (3.12.1), we have represented an arbitrary signal $x(n)$ into the weighted sum of unit impulses, we can now determine the response of any linear system to an input signal.

Let, $h(n, k)$ denote the response of the linear system to the shifted unit impulse $\delta(n-k)$ that is,

$$h(n, k) = \mathrm{T}\,[\delta(n-k)] \qquad \text{...(3.40)}$$

Then from the superposition property of a linear system, the response $y(n, k)$ of the linear system to the input $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$ is simply the weighted linear combination of these basic response. That is, with the input $x(n)$ to a linear system expressed in form of eqn. $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$, the output $y(n)$ can be expressed as,

$$y(n) = \mathrm{T}\,[x(n)]$$

$$= \mathrm{T}\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} [x(k)\,\mathrm{T}[\delta(n-k)]]$$

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n, k)} \qquad \text{...(3.41)}$$

Thus according to eqn. (3.41) if we know the response of a linear system to the set of shifted unit impulse, we can construct the response to an arbitrary input. If the response of the LTI system to the unit impulse $\delta(n)$ is denoted by $h(n)$, that is,

$$h(n) = \mathrm{T}[\delta(n)]$$

Then by the time invariance property of the response of the system to the delayed unit impulse $\delta(n-k)$; is,

$$h\,(n-k) = \mathrm{T}[\delta(n-k)]$$

Consequently eqn. (3.41) reduces to,

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)} \qquad \text{...(3.42)}$$

The eqn. (3.42) referred as the convolution sum or superposition sum and the operation on R.H.S. is known as the convolution of the sequence $x(n)$ and $h(n)$. We will represent the operation of convolution symbolically as,

$$\boxed{y(n) = x(n) * h(n)} \quad \ldots(3.43)$$

The operation of discrete-time convolution takes two sequences $x(n)$ and $h(n)$ and produces a third sequence $y(n)$.

3.13 THE CONVOLUTION PROCESS CAN BE SUMMARISED INTO THE FOLLOWING STEPS

Step 1. Choose an initial value of 'n', the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$, then $n = n_1 + n_2$, is a good choice. Then express both sequence in terms of the index k.

Step 2. *Folding.* Fold the $h(k)$ about the origin and obtain $h(-k)$.

Step 3. *Time shifting.* Shift the $h(-k)$ by n unit to right if n is positive and left if n is negative to obtain $h[-(k-n)] = h(n-k)$.

Step 4. *Multiplication.* Multiply $x(k)$ by $h(n-k)$ to obtain $w_n(k) = x(k)\,h(n-k)$.

Step 5. *Summation.* Sum all the values of the product $w(k)$ to obtain the value of output $y(n)$.

Step 6. Increment the index n, shift the sequence $h(n-k)$ to right by one sample and do step 4.

Step 7. Repeat step 6 until the sum of product is zero for all remaining values of n.

Problem 15. *Determine the output y(n) of a linear time invariant system with impulse response*

$$n(n) = \{\underset{\uparrow}{6}, 5, 4, 3, 2, 1\}$$

when the input is

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$$

Sol. $x(n)$ starts at $n_1 = 0$ and $h(n)$ starts at $n_2 = 0$.

Therefore, the starting value of $n = n_1 + n_2 = 0$.

Step 1. *Folding.* The folding sequence $h(-k)$ is illustrated in Fig. 3.14.

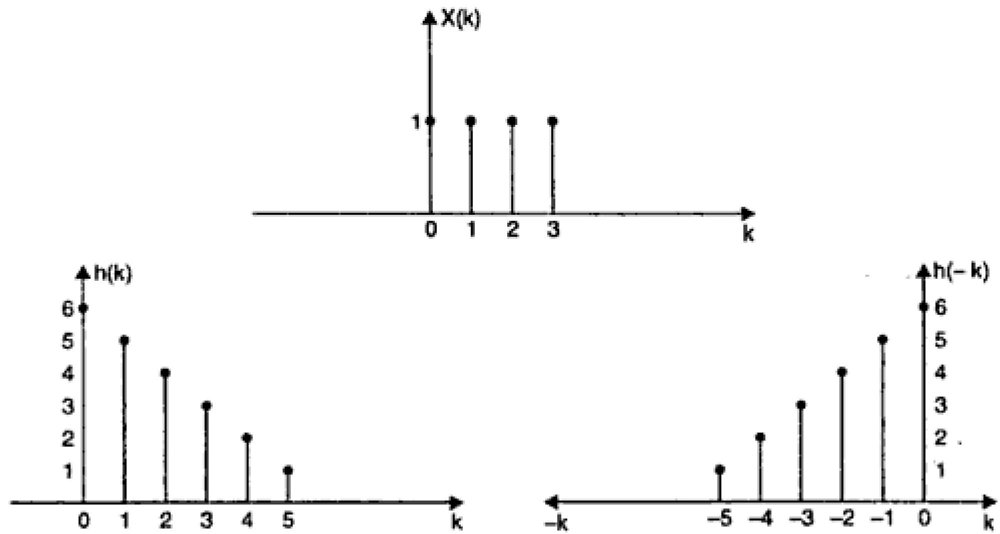
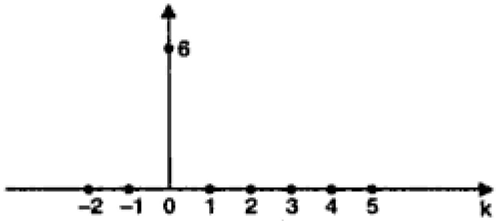


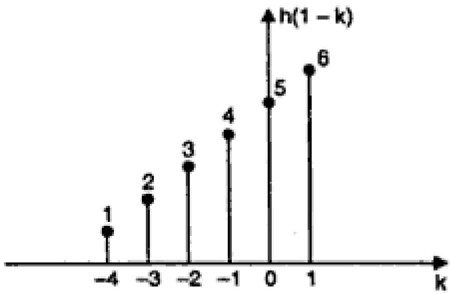
Fig. 3.14. Folding sequence.

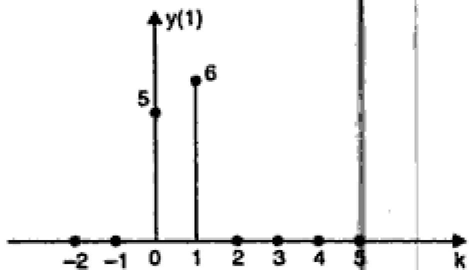
For n = 0, $$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k)$$

$$= x(0)\, h(0) = (1 \times 6) = 6$$



For n = 1, $$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

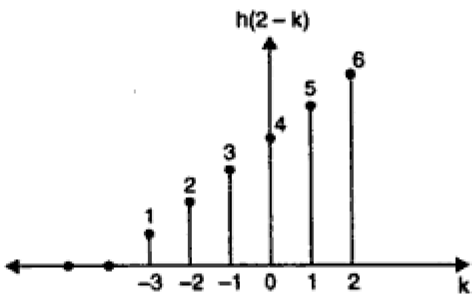


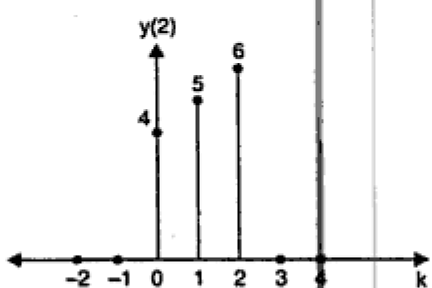


$$y(1) = x(0)\, h(1) + x(1)\, h(0)$$

$$= (1 \times 6) + (1 \times 5) = 11.$$

For n = 2, $$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k).$$



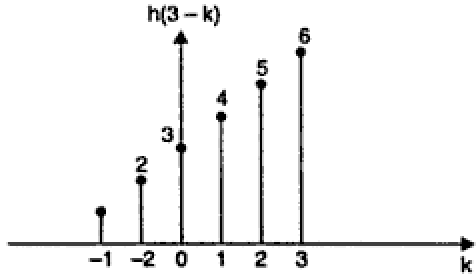


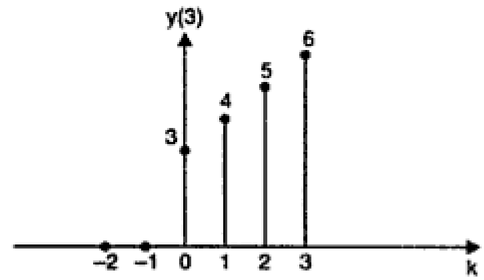
$$y(2) = x(0)\, h\,(2) + x(1)\, h(1) + x(2)\, h(0).$$

$$= (1 \times 4) + (1 \times 5) + (1 \times 6)$$

$$= 4 + 5 + 6 = 15.$$

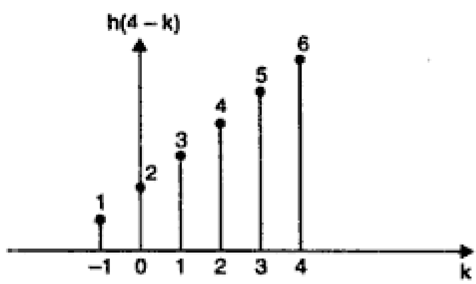
For n = 3, $y(3) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k)$

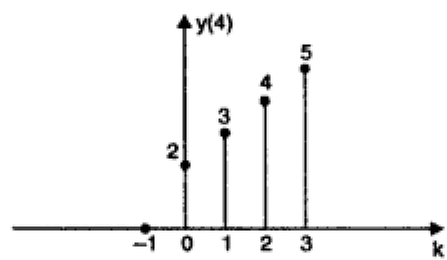




$$\begin{aligned} y(3) &= x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h(0) \\ &= (1 \times 3) + (1 \times 4) + (1 \times 5) + (1 \times 6) \\ &= 3 + 4 + 5 + 6 = 18 \end{aligned}$$

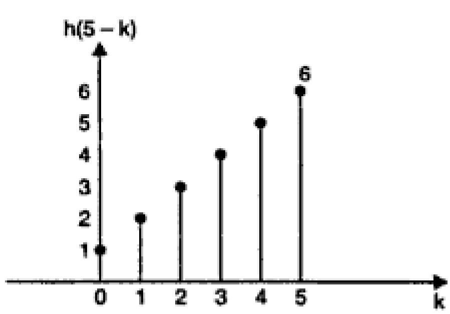
For n = 4, $y(4) = \sum_{k=-\infty}^{\infty} x(k)\, h(4-k)$

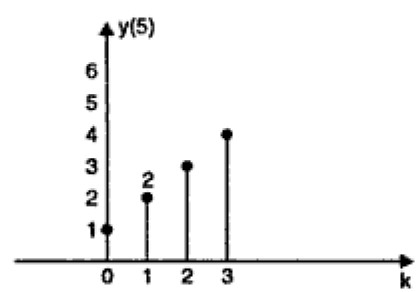




$$\begin{aligned} y(4) &= x(0)\,h(4) + x(1)\,h(3) + x(2)\,h(2) + x(3)\,h(1) + x(4)\,h(0) \\ &= (1 \times 2) + (1 \times 3) + (1 \times 4) + (1 \times 5) + (1 \times 6) \\ &= 2 + 3 + 4 + 5 = 24 \end{aligned}$$

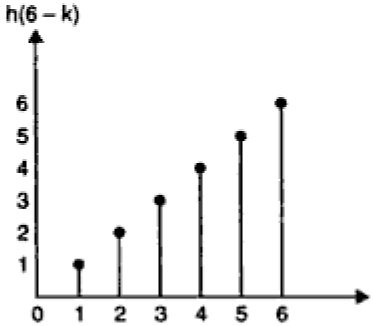
For n = 5, $y(5) = \sum_{k=-\infty}^{\infty} x(k)\, h(5-k)$

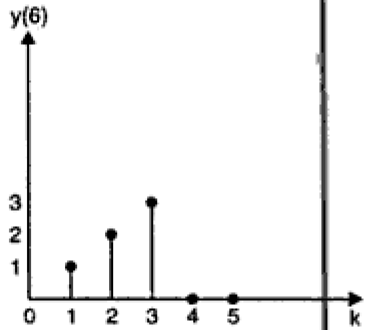




$$y(5) = x(0)\,h(5) + x(1)\,h(4) + x(2)\,h(3) + x(3)\,h(2) + x(4)\,h(1) + x(5)\,h(0)$$
$$= (1 \times 1) + (1 \times 2) + (1 \times 3) + (1 \times 4) + 0 + 0$$
$$= 1 + 2 + 3 + 4 = 10$$

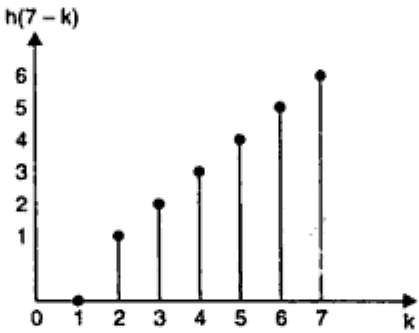
For n = 6, $y(6) = \sum_{k=-\infty}^{\infty} x(k)\,h(6-k)$

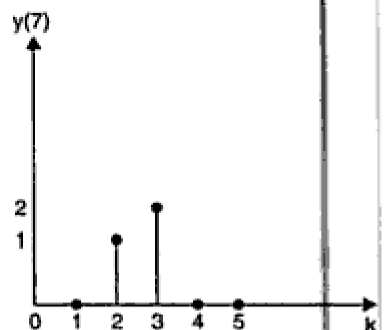




$$y(6) = x(0)\,h(6) + x(1)\,h(5) + x(2)\,h(4) + x(3)\,h(3)$$
$$+ x(4)\,h(2) + x(5)\,h(1) + x(6)\,h(0)$$
$$= 0 + (1 \times 1) + (1 \times 2) + (1 \times 3) + 0 + 0 + 0$$
$$= 1 + 2 + 3 = 6.$$

For n = 7, $y(7) = \sum_{k=-\infty}^{\infty} x(k)\,h(7-k)$

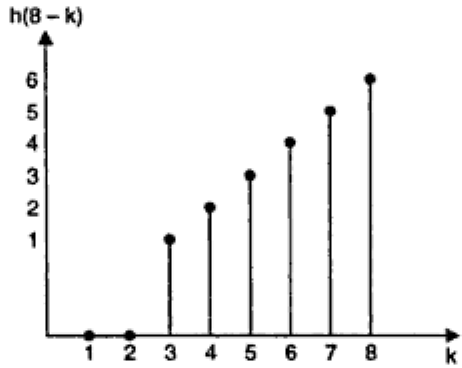


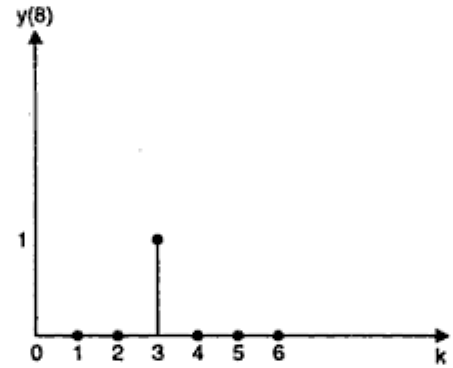


$$y(7) = x(0)\,h(7) + x(1)\,h(6) + x(2)\,h(5) + x(3)\,h(4)$$
$$+ x(4)\,h(3) + x(5)\,h(2) + x(6)\,h(1) + x(7)\,h(0)$$
$$= 0 + 0 + (1 \times 1) + (1 \times 2) + 0 + 0 + 0 + 0$$
$$= 1 + 2 = 3.$$

For n = 8, $y(8) = \sum_{k=-\infty}^{\infty} x(k)\,y(8-k)$

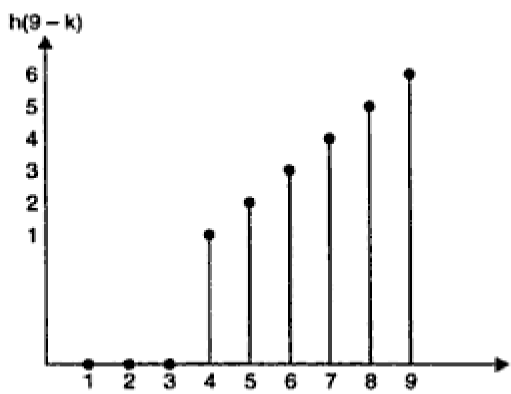
$$y(8) = x(0)\,h(8) + x(1)\,h(7) + x(2)\,h(6) + x(3)\,h(5)$$
$$+ x(4)\,h(4) + x(5)\,h(3) + x(6)\,h(2)$$
$$+ x(7)\,h(1) + x(8)\,h(0)$$

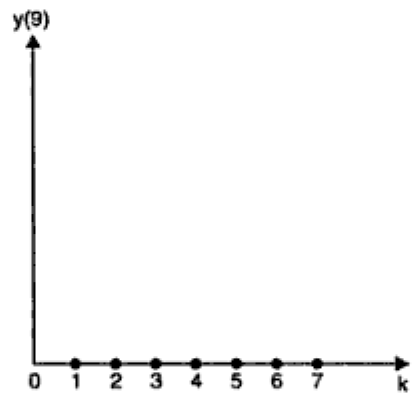




$$= 0 + 0 + 0 + (1 \times 1) + 0 + 0 + 0 + 0 + 0$$
$$= 1$$

For n = 9, $y(9) = \sum_{k=-\infty}^{\infty} x(k)\, h(9-k)$





$$\begin{aligned} y(9) &= x(0)\,h(9) + x(1)\,h(8) + x(2)\,h(7) + x(3)\,h(6) \\ &\quad + x(4)\,h(5) + x(5)\,h(4) + x(6)\,h(3) \\ &\quad + x(7)\,h(2) + x(8)\,h(1) + x(9)\,h(0). \\ &= 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0. \end{aligned}$$

Similarly, $y(-1) = 0$.

Now we summarize the entire response for $-\infty < n < \infty$ as below :

$$y(n) = \{0, 0, \underset{\uparrow}{6}, 11, 15, 18, 19, 10, 6, 3, 1, 0, 0\}.$$

EXERCISES

1. Determine the convolution $y(n)$ of the signals

$$x(n) = \begin{cases} 1 & 3 \le n \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(n) = \begin{cases} 1 & -4 \le n \le -3 \\ 0 & \text{elsewhere} \end{cases}$$

Ans. $y(n) = \{1, 2, 2, 2, 1\}$.

2. Find the convolution of the signals,

$$x(n) = 1 \quad n = -2, 0, 1$$
$$= 2 \quad n = -1$$
$$= 0 \quad \text{elsewhere.}$$

$n(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$ *i.e.*, $h(n) = \{1, -1, 1, -1\}$

Ans. $y(n) = \{1, 1, 0, 1, -2, 0, -1\}$.
$\uparrow$

3.14 PROPERTIES OF LINEAR TIME-INVARIANT SYSTEM

Since all linear-invariant systems are described by the convolution sum, the properties of this class of system are defined by the properties of discrete-time convolution.

(1) **Commutative.** The convolution operation is commutative

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

where, $$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(3.44)$$

and $$h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(m)\, x(n-m) \qquad ...(3.45)$$

In eqn. (3.44), the impulse response is folded on the other hand the input signal is folded in eqn. (3.46)

Commutative operation can be shown by applying a substitution of variables, with $m = n - k$.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{h=-\infty}^{\infty} h(m)\, x(n-m) \qquad ...(3.46)$$

(2) **Distributive (Parallel connection)**

The convolution operation is also distributive over addition *i.e.*,

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n) \qquad ...(3.47)$$

Distributive law implies that if we have two linear time invariant systems with impulse response $h_1(n)$ and $h_2(n)$ excited by the same input $x(n)$ then the sum of the two response identical to the response of an overall system with impulse response

$$h(n) = h_1(n) + h_2(n) \qquad ...(3.48)$$

Thus the overall system is viewed as parallel combination of two linear time-invariant system. The parallel operation is shown in Fig. 3.15

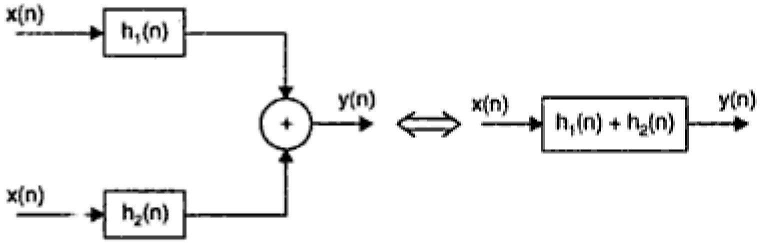


Fig. 3.15. Parallel operation.

(3) Associative [Cascade connection]

The associative property states that, when we are convolving three signals, we can convolve two of them and then convolve that result with the third signal.

Let us assume that,

$x(n)$, $h_1(n)$ and $h_2(n)$ are three arbitrary signals to be convolved, *i.e.*,

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)] \quad ...(3.49)$$

An interpretation of the associative property is illustrated in Fig. 3.16

Fig. 3.16. Associative property.

$$y(n) = w(n) * h_2(n) \qquad y(n) = x(n) * [h_1(n) + h_2(n)]$$

$$= [x(n) * h_1(n)] * h_2(n).$$

Problem 16. *Prove that* $x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$.

Sol. $x(n) * [h_1(n) * h_2(n)] = x(n) * \left[\sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k)\right].$

$$= \sum_{L=-\infty}^{\infty} x(l) \left[\sum_{k=-\infty}^{\infty} h_1(k)\, h_2(n-k-L)\right].$$

Let, $k - l = m$.

$$= \sum_{L=-\infty}^{\infty} x(l) \left[\sum_{m=-\infty}^{\infty} h_1(m-L)\, h_2(n-m)\right]$$

$$= \sum_{m=-\infty}^{\infty} \left[\sum_{L=-\infty}^{\infty} x(L)\, h_1(m-L)\right] h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} [x(m) * h(m)]\, h_2(n-m)$$

$$= [x(n) * h_1(n)] * h_2(n).$$

Problem 17. *Find the convolution of two finite duration sequences.*

$$h(n) = a^n u(n) \qquad \text{for all } n$$

$$x(n) = b^n u(n) \qquad \text{for all } n \qquad (i) \text{ when } a \neq b \ (ii) \text{ when } a = b.$$

Sol. The impulse response $h(n) = 0$ for $n < 0$, So the system is causal and $x(n) = 0$ for $n < 0$, hence the sequence is a causal sequence.

About the Book

This book deals with the analysis of Digital Signal Processing in a lucid and precise style. There are about 200 solved problems apart from exercises.

This book covers the latest syllabus prescribed by the Anna University for Electrical and Electronics engineering students. Exercise problems and review questions are included at the end of each chapter. All the above aspects should make this book extremely valuable for engineering students preparing for Anna University examinations as well as for practicing engineers.

About the Author

C. Ramesh Babu Durai graduated from Arulmigu Kalasalingam College of Engineering, Srivilliputhur and did his post graduate studies at Hindustan College of Engineering, Chennai. He is a faculty member of the department of Electrical and Electronics Engineering, Hindustan College of Engineering, Chennai.

He has more than six years of teaching experience and the college has honoured him by conferring on him "The Best Teacher Award". His field of interest includes Control Systems, Advanced Digital Signal Processing and Electromagnetic theory.

LAXMI PUBLICATIONS (P) LTD

ISBN 81-7008-736-8

9 788170 087366