Corrections to *Digital Communications,* 4th Edition

1. Page 31, Equation (2.1-54)
   First line: y_1 instead of y_2
   Second line: g_n instead of g_1

2. Page 163, Equation (4.2-30)

   should be: $s(t) = a_o/2 + \sum_{k=1}^{\infty}$

3. Page 163, Equation (4.2-31)

   should be: $a_k = (2/T)\int_0^T s(t) \cos 2\pi kt/T \; dt \; , \; k \geq 0$

   $b_k = (2/T)\int_0^T s(t) \sin 2\pi kt/T \; dt \; , \; k \geq 1$

4. Page 178, 7 lines from the top

   should be: sqrt (2ε) instead of ε sqrt (2)

5. Page 238, Equation (5.1-19)

   should be: $h(T-\tau)$ instead of $h(t-\tau)$

6. Page 238, two lines below Equation (5.1-20)

   should be: $y_n^2(T)$ instead of $y_n^2(t)$

7. Page 244, Equation (5.1 – 45)

   should be: $m = 1,2,\ldots M$

8. Page 245, Equation (5.1-48)

   should be: : sqrt (ε_b) instead of sqrt (ε_n)

9. Page 309, Equation (5.4-39)

R_1 sqrt ($2\mathcal{E}_s/N_0$) instead of sqrt ($2\mathcal{E}_s R_1/N_0$)

10. Page 318, Equation (5.5-17)

add the term: $-(N_0)_{dBW/Hz}$

11. Page 366, Equation (6.4-3)

Replace + sign with – sign in the second term of the summation

12. Page 367, Equation (6.4-6)

Replace + sign with – sign in the second term of the summation

13. Page 367, Equations (6.4-8) and (6.4-9)

add the subscript L to the log-likelihood function

14. Page 422, lines 2 and 3 above Equation (8.1-14)

delete the phrase "no more than"

15. Page 468, 12 lines from the top and 5 lines from the bottom

should be: $b <$ instead of $b \leq$

16. Page 491, Figure 8.2-15

solid line corresponds to soft-decision decoding
broken line corresponds to hard-decision decoding

17. Page 500, Equation (8.2-41)

In the denominator, M_k should be M_j and M_j should be M_J

18. Page 591, Figure P9.9

The lower shaping filter in the modulator and demodulator,

q(t) should have a "hat" on it

19. Page 609, 6 lines above Equation (10.1-34)

$\varepsilon_{k+1-L-1}$ should be $\varepsilon_{k+l-L-1}$

20. Page 646, Figure 10.3-5

delete the “hat” from I(z)

21. Page 651, 4 lines from the top

replace “over” with “about”

22. Page 651, 2 lines above Section 10.6

“Turob” should be “Turbo”

23. Page 673, Figure 11.1-6

Lower delay line elements: z^1 should be z^{-1}

24. Page 750, Figure 13.2-8

Replace “adders” with “multipliers”

25. Page 752, Figure 13.2-9

Replace “adders” with multipliers”

26. Page 856, Equation (14.6-5)

Replace K with k

27. Page 885, Figure 14.7-7

The “Input” should be 02310

28. Page 894, Problem 14.16

$r_1 = h_1 s_1 + h_2 s_2 + n_1$

$r_2 = h_1 s_2^* + h_2 s_1^* + n_2$

29. Page 895

Delete 2^k from the expression on the error probability

30. Page 915, top of page

(15.47) should be (15.3-47)

31. Page 925, 6 lines form top

T_0 should be T_p

32. Page 935

a) top of page:

$r_1 = b_1$ sqrt$(\varepsilon_1) + b_2\rho$ sqrt $(\varepsilon_2) + n_1$

$r_2 = b_1\rho$ sqrt$(\varepsilon_1) + b_2$ sqrt $(\varepsilon_2) + n_2$

b) Problem 15.8, last equation

delete factor of 1/2

c) Problem 15.9, first equation

delete comma after b_2=1

33. Page 936, first equation at top of page, second term

should be:

$$\ln \cosh \left\{ [r_2 \text{ sqrt } (\varepsilon_2) - b_1\rho \text{ sqrt } (\varepsilon_1\varepsilon_2)]/N_0 \right\}$$

34. Page 936, second equation from top of page

divide each of the arguments in the cosh function by N_0

35. Page 936, Problem 15.10

should be $\eta_k = [\qquad]^2$

36. Page 936, Problem 15.11

the last term in the equation should be:

$$(1/2)\, Q \left\{ \text{sqrt}\left[\frac{\varepsilon_1 + \varepsilon_2 - 2|\rho|\ \text{sqrt}\,(\varepsilon_1 \varepsilon_2)}{N_0/2} \right] \right\}$$

CHAPTER 2

Problem 2.1 :

$$P(A_i) = \sum_{j=1}^{3} P(A_i, B_j), i = 1, 2, 3, 4$$

Hence :

$$P(A_1) = \sum_{j=1}^{3} P(A_1, B_j) = 0.1 + 0.08 + 0.13 = 0.31$$

$$P(A_2) = \sum_{j=1}^{3} P(A_2, B_j) = 0.05 + 0.03 + 0.09 = 0.17$$

$$P(A_3) = \sum_{j=1}^{3} P(A_3, B_j) = 0.05 + 0.12 + 0.14 = 0.31$$

$$P(A_4) = \sum_{j=1}^{3} P(A_4, B_j) = 0.11 + 0.04 + 0.06 = 0.21$$

Similarly :

$$P(B_1) = \sum_{i=1}^{4} P(A_i, B_1) = 0.10 + 0.05 + 0.05 + 0.11 = 0.31$$

$$P(B_2) = \sum_{i=1}^{4} P(A_i, B_2) = 0.08 + 0.03 + 0.12 + 0.04 = 0.27$$

$$P(B_3) = \sum_{i=1}^{4} P(A_i, B_3) = 0.13 + 0.09 + 0.14 + 0.06 = 0.42$$

Problem 2.2 :

The relationship holds for $n = 2$ (2-1-34) : $p(x_1, x_2) = p(x_2|x_1)p(x_1)$
Suppose it holds for $n = k$, i.e : $p(x_1, x_2, ..., x_k) = p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \ ...p(x_1)$
Then for $n = k + 1$:

$$\begin{aligned} p(x_1, x_2, ..., x_k, x_{k+1}) &= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k, x_{k-1} ..., x_1) \\ &= p(x_{k+1}|x_k, x_{k-1}, ..., x_1)p(x_k|x_{k-1}, ..., x_1)p(x_{k-1}|x_{k-2}, ..., x_1) \ ...p(x_1) \end{aligned}$$

Hence the relationship holds for $n = k + 1$, and by induction it holds for any n.

Problem 2.3 :

Following the same procedure as in example 2-1-1, we prove :

$$p_Y(y) = \frac{1}{|a|} p_X\left(\frac{y-b}{a}\right)$$

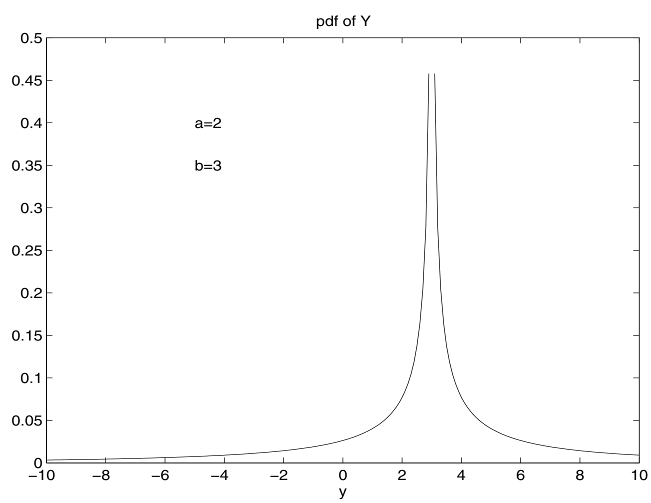
Problem 2.4 :

Relationship (2-1-44) gives :

$$p_Y(y) = \frac{1}{3a\left[(y-b)/a\right]^{2/3}} p_X\left[\left(\frac{y-b}{a}\right)^{1/3}\right]$$

X is a gaussian r.v. with zero mean and unit variance : $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
Hence :

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}\left[(y-b)/a\right]^{2/3}} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$



Problem 2.5 :

(a) Since (X_r, X_i) are statistically independent :

$$p_{\mathbf{X}}(x_r, x_i) = p_X(x_r) p_X(x_i) = \frac{1}{2\pi\sigma^2} e^{-\left(x_r^2 + x_i^2\right)/2\sigma^2}$$

Also :

$$Y_r + jY_i = (X_r + X_i)e^{j\phi} \Rightarrow$$

$$X_r + X_i = (Y_r + jY_i)\, e^{-j\phi} = Y_r \cos\phi + Y_i \sin\phi + j(-Y_r \sin\phi + Y_i \cos\phi) \Rightarrow$$

$$\left\{ \begin{array}{ll} X_r = & Y_r \cos\phi + Y_i \sin\phi \\ X_i = & -Y_r \sin\phi + Y_i \cos\phi \end{array} \right\}$$

The Jacobian of the above transformation is :

$$J = \left| \begin{array}{cc} \frac{\partial X_r}{\partial Y_r} & \frac{\partial X_i}{\partial Y_r} \\ \frac{\partial Xr}{\partial Y_i} & \frac{\partial X_i}{\partial Y_i} \end{array} \right| = \left| \begin{array}{cc} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{array} \right| = 1$$

Hence, by (2-1-55) :

$$\begin{array}{rcl} p_{\mathbf{Y}}(y_r, y_i) & = & p_{\mathbf{X}}((Y_r \cos\phi + Y_i \sin\phi)\,, (-Y_r \sin\phi + Y_i \cos\phi)) \\ & = & \frac{1}{2\pi\sigma^2} e^{-\left(y_r^2 + y_i^2\right)/2\sigma^2} \end{array}$$

(b) $\mathbf{Y} = \mathbf{AX}$ and $\mathbf{X} = \mathbf{A^{-1}Y}$

Now, $p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\mathbf{x}'\mathbf{x}/2\sigma^2}$ (the covariance matrix $\mathbf{M}$ of the random variables $x_1, ..., x_n$ is $\mathbf{M} = \sigma^2\mathbf{I}$, since they are i.i.d) and $J = 1/|\det(\mathbf{A})|$. Hence :

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{|\det(\mathbf{A})|} e^{-\mathbf{y}'(\mathbf{A}^{-1})'\mathbf{A}^{-1}\mathbf{y}/2\sigma^2}$$

For the pdf's of X and Y to be identical we require that :

$$|\det(\mathbf{A})| = 1 \; and \; (\mathbf{A}^{-1})'\mathbf{A}^{-1} = \mathbf{I} \implies \mathbf{A}^{-1} = \mathbf{A}'$$

Hence, $\mathbf{A}$ must be a unitary (orthogonal) matrix .

Problem 2.6 :

(a)

$$\psi_Y(jv) = E\left[e^{jvY}\right] = E\left[e^{jv\sum_{i=1}^n x_i}\right] = E\left[\prod_{i=1}^n e^{jvx_i}\right] = \prod_{i=1}^n E\left[e^{jvX}\right] = \left(\psi_X(e^{jv})\right)^n$$

But,

$$p_X(x) = p\delta(x-1) + (1-p)\delta(x) \Rightarrow \psi_X(e^{jv}) = 1 + p + pe^{jv}$$

$$\Rightarrow \psi_Y(jv) = \left(1 + p + pe^{jv}\right)^n$$

(b)

$$E(Y) = -j\frac{d\psi_Y(jv)}{dv}|_{v=0} = -jn(1-p+pe^{jv})^{n-1}jpe^{jv}|_{v=0} = np$$

and

$$E(Y^2) = -\frac{d^2\psi_Y(jv)}{d^2v}|_{v=0} = -\frac{d}{dv}\left[jn(1-p+pe^{jv})^{n-1}pe^{jv}\right]_{v=0} = np + np(n-1)p$$

$$\Rightarrow E(Y^2) = n^2p^2 + np(1-p)$$

Problem 2.7 :

$$\psi(jv_1, jv_2, jv_3, jv_4) = E\left[e^{j(v_1x_1+v_2x_2+v_3x_3+v_4x_4)}\right]$$

$$E\left(X_1X_2X_3X_4\right) = (-j)^4\frac{\partial^4\psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1\partial v_2\partial v_3\partial v_4}|_{v_1=v_2=v_3=v_4=0}$$

From (2-1-151) of the text, and the zero-mean property of the given rv's :

$$\psi(j\mathbf{v}) = e^{-\frac{1}{2}\mathbf{v'Mv}}$$

where $\mathbf{v} = [v_1, v_2, v_3, v_4]'$, $\mathbf{M} = [\mu_{ij}]$.
We obtain the desired result by bringing the exponent to a scalar form and then performing quadruple differentiation. We can simplify the procedure by noting that :

$$\frac{\partial\psi(j\mathbf{v})}{\partial v_i} = -\mu_{\mathbf{i}}'\mathbf{v}e^{-\frac{1}{2}\mathbf{v'Mv}}$$

where $\mu_{\mathbf{i}}' = [\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}]$. Also note that :

$$\frac{\partial\mu_{\mathbf{j}}'\mathbf{v}}{\partial v_i} = \mu_{ij} = \mu_{ji}$$

Hence :

$$\frac{\partial^4\psi(jv_1, jv_2, jv_3, jv_4)}{\partial v_1\partial v_2\partial v_3\partial v_4}|_{\mathbf{V=0}} = \mu_{12}\mu_{34} + \mu_{23}\mu_{14} + \mu_{24}\mu_{13}$$

Problem 2.8 :

For the central chi-square with n degress of freedom :

$$\psi(jv) = \frac{1}{(1-j2v\sigma^2)^{n/2}}$$

Now :

$$\frac{d\psi(jv)}{dv} = \frac{jn\sigma^2}{(1-j2v\sigma^2)^{n/2+1}} \Rightarrow E(Y) = -j\frac{d\psi(jv)}{dv}|_{v=0} = n\sigma^2$$

$$\frac{d^2\psi(jv)}{dv^2} = \frac{-2n\sigma^4(n/2+1)}{(1-j2v\sigma^2)^{n/2+2}} \Rightarrow E\left(Y^2\right) = -\frac{d^2\psi(jv)}{dv^2}|_{v=0} = n(n+2)\sigma^2$$

The variance is $\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 2n\sigma^4$

For the non-central chi-square with n degrees of freedom :

$$\psi(jv) = \frac{1}{(1-j2v\sigma^2)^{n/2}} e^{jvs^2/\left(1-j2v\sigma^2\right)}$$

where by definition : $s^2 = \sum_{i=1}^n m_i^2$.

$$\frac{d\psi(jv)}{dv} = \left[\frac{jn\sigma^2}{(1-j2v\sigma^2)^{n/2+1}} + \frac{js^2}{(1-j2v\sigma^2)^{n/2+2}}\right] e^{jvs^2/\left(1-j2v\sigma^2\right)}$$

Hence, $E(Y) = -j\frac{d\psi(jv)}{dv}|_{v=0} = n\sigma^2 + s^2$

$$\frac{d^2\psi(jv)}{dv^2} = \left[\frac{-n\sigma^4(n+2)}{(1-j2v\sigma^2)^{n/2+2}} + \frac{-s^2(n+4)\sigma^2 - ns^2\sigma^2}{(1-j2v\sigma^2)^{n/2+3}} + \frac{-s^4}{(1-j2v\sigma^2)^{n/2+4}}\right] e^{jvs^2/\left(1-j2v\sigma^2\right)}$$

Hence,

$$E\left(Y^2\right) = -\frac{d^2\psi(jv)}{dv^2}|_{v=0} = 2n\sigma^4 + 4s^2\sigma^2 + \left(n\sigma^2 + s^2\right)$$

and

$$\sigma_Y^2 = E\left(Y^2\right) - [E(Y)]^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

Problem 2.9 :

The Cauchy r.v. has : $p(x) = \frac{a/\pi}{x^2+a^2}, -\infty < x < \infty$ **(a)**

$$E(X) = \int_{-\infty}^{\infty} xp(x)dx = 0$$

since $p(x)$ is an even function.

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{x^2+a^2}dx$$

Note that for large x, $\frac{x^2}{x^2+a^2} \rightarrow 1$ (i.e non-zero value). Hence,

$$E\left(X^2\right) = \infty, \sigma^2 = \infty$$

(b)

$$\psi(jv) = E\left(^{jvX}\right) = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2+a^2} e^{jvx} dx = \int_{-\infty}^{\infty} \frac{a/\pi}{(x+ja)(x-ja)} e^{jvx} dx$$

This integral can be evaluated by using the residue theorem in complex variable theory. Then, for $v \geq 0$:

$$\psi(jv) = 2\pi j \left(\frac{a/\pi}{x+ja} e^{jvx} \right)_{x=ja} = e^{-av}$$

For $v < 0$:

$$\psi(jv) = -2\pi j \left(\frac{a/\pi}{x-ja} e^{jvx} \right)_{x=-ja} = e^{av} v$$

Therefore :

$$\psi(jv) = e^{-a|v|}$$

Note: an alternative way to find the characteristic function is to use the Fourier transform relationship between $p(x), \psi(jv)$ and the Fourier pair :

$$e^{-b|t|} \leftrightarrow \frac{1}{\pi} \frac{c}{c^2+f^2}, \ c = b/2\pi, \ f = 2\pi v$$

Problem 2.10 :

(a) $Y = \frac{1}{n}\sum_{i=1}^{n} X_i, \ \psi_{X_i}(jv) = e^{-a|v|}$

$$\psi_Y(jv) = E\left[e^{jv\frac{1}{n}\sum_{i=1}^{n} X_i}\right] = \prod_{i=1}^{n} E\left[e^{j\frac{v}{n}X_i}\right] = \prod_{i=1}^{n} \psi_{X_i}(jv/n) = \left[e^{-a|v|/n}\right]^n = e^{-a|v|}$$

(b) Since $\psi_Y(jv) = \psi_{X_i}(jv) \Rightarrow p_Y(y) = p_{X_i}(x_i) \Rightarrow p_Y(y) = \frac{a/\pi}{y^2+a^2}$.

(c) As $n \to \infty$, $p_Y(y) = \frac{a/\pi}{y^2+a^2}$, which is not Gaussian ; hence, the central limit theorem does not hold. The reason is that the Cauchy distribution does not have a finite variance.

Problem 2.11 :

We assume that $x(t), y(t), z(t)$ are real-valued stochastic processes. The treatment of complex-valued processes is similar.

(a)

$$\phi_{zz}(\tau) = E\left\{[x(t+\tau)+y(t+\tau)]\,[x(t)+y(t)]\right\} = \phi_{xx}(\tau)+\phi_{xy}(\tau)+\phi_{yx}(\tau)+\phi_{yy}(\tau)$$

(b) When $x(t), y(t)$ are uncorrelated :

$$\phi_{xy}(\tau) = E\left[x(t+\tau)y(t)\right] = E\left[x(t+\tau)\right]E\left[y(t)\right] = m_x m_y$$

Similarly :

$$\phi_{yx}(\tau) = m_x m_y$$

Hence :

$$\phi_{zz}(\tau) = \phi_{xx}(\tau) + \phi_{yy}(\tau) + 2m_x m_y$$

(c) When $x(t), y(t)$ are uncorrelated and have zero means :

$$\phi_{zz}(\tau) = \phi_{xx}(\tau) + \phi_{yy}(\tau)$$

Problem 2.12 :

The power spectral density of the random process $x(t)$ is :

$$\Phi_{xx}(f) = \int_{-\infty}^{\infty} \phi_{xx}(\tau) e^{-j2\pi f\tau} d\tau = N_0/2.$$

The power spectral density at the output of the filter will be :

$$\Phi_{yy}(f) = \Phi_{xx}(f)|H(f)|^2 = \frac{N_0}{2}|H(f)|^2$$

Hence, the total power at the output of the filter will be :

$$\phi_{yy}(\tau = 0) = \int_{-\infty}^{\infty} \Phi_{yy}(f) df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}(2B) = N_0 B$$

Problem 2.13 :

$$\mathbf{M}_X = E\left[(\mathbf{X} - \mathbf{m}_x)(\mathbf{X} - \mathbf{m}_x)'\right], \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}, \ \mathbf{m}_x \text{ is the corresponding vector of mean values.}$$

Then :

$$\begin{aligned}\mathbf{M}_Y &= E\left[(\mathbf{Y}-\mathbf{m}_y)(\mathbf{Y}-\mathbf{m}_y)'\right] \\ &= E\left[\mathbf{A}(\mathbf{X}-\mathbf{m}_x)(\mathbf{A}(\mathbf{X}-\mathbf{m}_x))'\right] \\ &= E\left[\mathbf{A}(\mathbf{X}-\mathbf{m}_x)(\mathbf{X}-\mathbf{m}_x)'\mathbf{A}'\right] \\ &= \mathbf{A}\mathbf{E}\left[(\mathbf{X}-\mathbf{m_x})(\mathbf{X}-\mathbf{m_x})'\right]\mathbf{A}' \\ &= \mathbf{A}\mathbf{M_x}\mathbf{A}'\end{aligned}$$

Hence :

$$\mathbf{M}_Y = \begin{bmatrix} \mu_{11} & 0 & \mu_{11}+\mu_{13} \\ 0 & 4\mu_{22} & 0 \\ \mu_{11}+\mu_{31} & 0 & \mu_{11}+\mu_{13}+\mu_{31}+\mu_{33} \end{bmatrix}$$

Problem 2.14 :

$Y(t) = X^2(t),\ \phi_{xx}(\tau) = E\left[x(t+\tau)x(t)\right]$

$$\phi_{yy}(\tau) = E\left[y(t+\tau)y(t)\right] = E\left[x^2(t+\tau)x^2(t)\right]$$

Let $X_1 = X_2 = x(t),\ \ X_3 = X_4 = x(t+\tau)$. Then, from problem 2.7 :

$$E\left(X_1X_2X_3X_4\right) = E\left(X_1X_2\right)E\left(X_3X_4\right) + E\left(X_1X_3\right)E\left(X_2X_4\right) + E\left(X_1X_4\right)E\left(X_2X_3\right)$$

Hence :

$$\phi_{yy}(\tau) = \phi_{xx}^2(0) + 2\phi_{xx}^2(\tau)$$

Problem 2.15 :

$p_R(r) = \frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m r^{2m-1}e^{-mr^2/\Omega},\quad X = \frac{1}{\sqrt{\Omega}}R$

We know that : $p_X(x) = \frac{1}{1/\sqrt{\Omega}}p_R\left(\frac{x}{1/\sqrt{\Omega}}\right)$.

Hence :

$$p_X(x) = \frac{1}{1/\sqrt{\Omega}}\frac{2}{\Gamma(m)}\left(\frac{m}{\Omega}\right)^m\left(x\sqrt{\Omega}\right)^{2m-1}e^{-m(x\sqrt{\Omega})^2/\Omega} = \frac{2}{\Gamma(m)}m^m x^{2m-1}e^{-mx^2}$$

Problem 2.16 :

The transfer function of the filter is :

$$H(f) = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{j\omega RC+1} = \frac{1}{j2\pi fRC+1}$$

(a)

$$\Phi_{xx}(f)=\sigma^2 \Rightarrow \Phi_{yy}(f)=\Phi_{xx}(f)\left|H(f)\right|^2=\frac{\sigma^2}{\left(2\pi RC\right)^2 f^2+1}$$

(b)

$$\phi_{yy}(\tau)=F^{-1}\{\Phi_{xx}(f)\}=\frac{\sigma^2}{RC}\int_{-\infty}^{\infty}\frac{\frac{1}{RC}}{(\frac{1}{RC})^2+(2\pi f)^2}e^{j2\pi f\tau}df$$

Let : $a=RC,\ \ v=2\pi f$. Then :

$$\phi_{yy}(\tau)=\frac{\sigma^2}{2RC}\int_{-\infty}^{\infty}\frac{a/\pi}{a^2+v^2}e^{jv\tau}dv=\frac{\sigma^2}{2RC}e^{-a|\tau|}=\frac{\sigma^2}{2RC}e^{-|\tau|/RC}$$

where the last integral is evaluated in the same way as in problem P-2.9 . Finally :

$$E\left[Y^2(t)\right]=\phi_{yy}(0)=\frac{\sigma^2}{2RC}$$

Problem 2.17 :

If $\Phi_X(f)=0$ for $|f|>W$, then $\Phi_X(f)e^{-j2\pi fa}$ is also bandlimited. The corresponding autocorrelation function can be represented as (remember that $\Phi_X(f)$ is deterministic) :

$$\phi_X(\tau-a)=\sum_{n=-\infty}^{\infty}\phi_X(\frac{n}{2W}-a)\frac{\sin 2\pi W\left(\tau-\frac{n}{2W}\right)}{2\pi W\left(\tau-\frac{n}{2W}\right)} \qquad (1)$$

Let us define :

$$\hat{X}(t)=\sum_{n=-\infty}^{\infty}X(\frac{n}{2W})\frac{\sin 2\pi W\left(t-\frac{n}{2W}\right)}{2\pi W\left(t-\frac{n}{2W}\right)}$$

We must show that :

$$E\left[|X(t)-\hat{X}(t)|^2\right]=0$$

or

$$E\left[\left(X(t)-\hat{X}(t)\right)\left(X(t)-\sum_{m=-\infty}^{\infty}X(\frac{m}{2W})\frac{\sin 2\pi W\left(t-\frac{m}{2W}\right)}{2\pi W\left(t-\frac{m}{2W}\right)}\right)\right]=0 \qquad (2)$$

First we have :

$$E\left[\left(X(t)-\hat{X}(t)\right)X(\frac{m}{2W})\right]=\phi_X(t-\frac{m}{2W})-\sum_{n=-\infty}^{\infty}\phi_X(\frac{n-m}{2W})\frac{\sin 2\pi W\left(t-\frac{n}{2W}\right)}{2\pi W\left(t-\frac{n}{2W}\right)}$$

But the right-hand-side of this equation is equal to zero by application of (1) with $a = m/2W$. Since this is true for any m, it follows that $E\left[\left(X(t) - \hat{X}(t)\right)\hat{X}(t)\right] = 0$. Also

$$E\left[\left(X(t) - \hat{X}(t)\right)X(t)\right] = \phi_X(0) - \sum_{n=-\infty}^{\infty} \phi_X(\frac{n}{2W} - t)\frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{2\pi W\left(t - \frac{n}{2W}\right)}$$

Again, by applying (1) with $a = t$ anf $\tau = t$, we observe that the right-hand-side of the equation is also zero. Hence (2) holds.

Problem 2.18 :

$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-t^2/2}dt = P\left[N \geq x\right]$, where N is a Gaussian r.v with zero mean and unit variance. From the Chernoff bound :

$$P\left[N \geq x\right] \leq e^{-\hat{v}x}E\left(e^{\hat{v}N}\right) \qquad (1)$$

where $\hat{v}$ is the solution to :

$$E\left(Ne^{vN}\right) - xE\left(e^{vN}\right) = 0 \qquad (2)$$

Now :

$$\begin{aligned} E\left(e^{vN}\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{vt}e^{-t^2/2}dt \\ &= e^{v^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-(t-v)^2/2}dt \\ &= e^{v^2/2} \end{aligned}$$

and

$$E\left(Ne^{vN}\right) = \frac{d}{dv}E\left(e^{vN}\right) = ve^{v^2/2}$$

Hence (2) gives :

$$\hat{v} = x$$

and then :

$$(1) \Rightarrow Q(x) \leq e^{-x^2}e^{x^2/2} \Rightarrow Q(x) \leq e^{-x^2/2}$$

Problem 2.19 :

Since $H(0) = \sum_{-\infty}^{\infty} h(n) = 0 \Rightarrow m_y = m_x H(0) = 0$

The autocorrelation of the output sequence is

$$\phi_{yy}(k) = \sum_i \sum_j h(i)h(j)\phi_{xx}(k-j+i) = \sigma_x^2 \sum_{i=-\infty}^{\infty} h(i)h(k+i)$$

where the last equality stems from the autocorrelation function of $X(n)$:

$$\phi_{xx}(k-j+i) = \sigma_x^2 \delta(k-j+i) = \left\{ \begin{array}{ll} \sigma_x^2, & j=k+i \\ 0, & o.w. \end{array} \right\}$$

Hence, $\phi_{yy}(0) = 6\sigma_x^2,\ \phi_{yy}(1) = \phi_{yy}(-1) = -4\sigma_x^2,\ \phi_{yy}(2) = \phi_{yy}(-2) = \sigma_x^2,\ \phi_{yy}(k) = 0$ otherwise. Finally, the frequency response of the discrete-time system is :

$$\begin{array}{rcl} H(f) & = & \sum_{-\infty}^{\infty} h(n)e^{-j2\pi fn} \\ & = & 1 - 2e^{-j2\pi f} + e^{-j4\pi f} \\ & = & \left(1 - e^{-j2\pi f}\right)^2 \\ & = & e^{-j2\pi f}\left(e^{j\pi f} - e^{-j\pi f}\right)^2 \\ & = & -4e^{-j\pi f} \sin^2 \pi f \end{array}$$

which gives the power density spectrum of the output :

$$\Phi_{yy}(f) = \Phi_{xx}(f)|H(f)|^2 = \sigma_x^2 \left[16 \sin^4 \pi f\right] = 16\sigma_x^2 \sin^4 \pi f$$

Problem 2.20 :

$$\phi(k) = \left(\frac{1}{2}\right)^{|k|}$$

The power density spectrum is

$$\begin{array}{rcl} \Phi(f) & = & \sum_{k=-\infty}^{\infty} \phi(k)e^{-j2\pi fk} \\ & = & \sum_{k=-\infty}^{-1} \left(\frac{1}{2}\right)^{-k} e^{-j2\pi fk} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{-j2\pi fk} \\ & = & \sum_{k=0}^{\infty}(\frac{1}{2}e^{j2\pi fk})^k + \sum_{k=0}^{\infty}(\frac{1}{2}e^{-j2\pi f})^k - 1 \\ & = & \frac{1}{1-e^{j2\pi f}/2} + \frac{1}{1-e^{-j2\pi f}/2} - 1 \\ & = & \frac{2-\cos 2\pi f}{5/4-\cos 2\pi f} - 1 \\ & = & \frac{3}{5-4\cos 2\pi f} \end{array}$$

Problem 2.21 :

We will denote the discrete-time process by the subscript d and the continuous-time (analog) process by the subscript a. Also, f will denote the analog frequency and f_d the discrete-time frequency.

(a)

$$\begin{aligned}\phi_d(k) &= E\left[X^*(n)X(n+k)\right] \\ &= E\left[X^*(nT)X(nT+kT)\right] \\ &= \phi_a(kT)\end{aligned}$$

Hence, the autocorrelation function of the sampled signal is equal to the sampled autocorrelation function of $X(t)$.

(b)

$$\begin{aligned}\phi_d(k) &= \phi_a(kT) = \int_{-\infty}^{\infty}\Phi_a(F)e^{j2\pi fkT}df \\ &= \sum_{l=-\infty}^{\infty}\int_{(2l-1)/2T}^{(2l+1)/2T}\Phi_a(F)e^{j2\pi fkT}df \\ &= \sum_{l=-\infty}^{\infty}\int_{-1/2T}^{1/2T}\Phi_a(f+\tfrac{l}{T})e^{j2\pi FkT}df \\ &= \int_{-1/2T}^{1/2T}\left[\sum_{l=-\infty}^{\infty}\Phi_a(f+\tfrac{l}{T})\right]e^{j2\pi FkT}df\end{aligned}$$

Let $f_d = fT$. Then :

$$\phi_d(k) = \int_{-1/2}^{1/2}\left[\frac{1}{T}\sum_{l=-\infty}^{\infty}\Phi_a((f_d+l)/T)\right]e^{j2\pi f_d k}df_d \qquad (1)$$

We know that the autocorrelation function of a discrete-time process is the inverse Fourier transform of its power spectral density

$$\phi_d(k) = \int_{-1/2}^{1/2}\Phi_d(f_d)e^{j2\pi f_d k}df_d \qquad (2)$$

Comparing (1),(2) :

$$\Phi_d(f_d) = \frac{1}{T}\sum_{l=-\infty}^{\infty}\Phi_a(\frac{f_d+l}{T}) \qquad (3)$$

(c) From (3) we conclude that :

$$\Phi_d(f_d) = \frac{1}{T}\Phi_a(\frac{f_d}{T})$$

iff :

$$\Phi_a(f) = 0, \quad \forall\ f : |f| > 1/2T$$

Otherwise, the sum of the shifted copies of Φ_a (in (3)) will overlap and aliasing will occur.

Problem 2.22 :

(a)

$$\begin{aligned}\phi_a(\tau) &= \int_{-\infty}^{\infty} \Phi_a(f) e^{j2\pi f\tau} df \\ &= \int_{-W}^{W} e^{j2\pi f\tau} df \\ &= \frac{\sin 2\pi W\tau}{\pi\tau}\end{aligned}$$

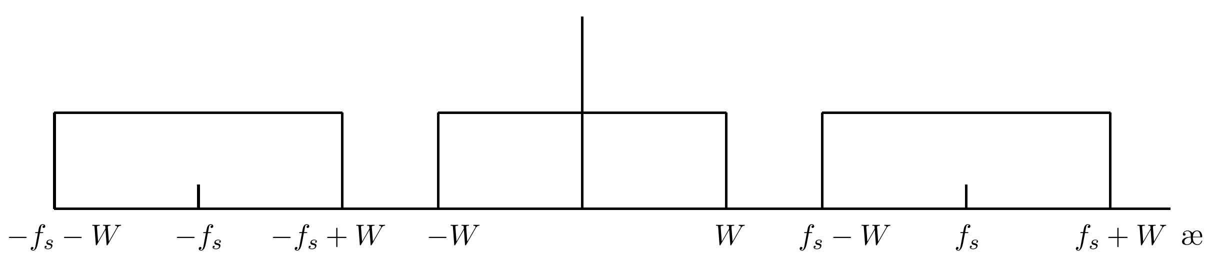
By applying the result in problem 2.21, we have

$$\phi_d(k) = f_a(kT) = \frac{\sin 2\pi WkT}{\pi kT}$$

(b) If $T = \frac{1}{2W}$, then :

$$\phi_d(k) = \left\{ \begin{array}{cc} 2W = 1/T, & \text{k} = 0 \\ 0, & \text{otherwise} \end{array} \right\}$$

Thus, the sequence $X(n)$ is a white-noise sequence. The fact that this is the minimum value of T can be shown from the following figure of the power spectral density of the sampled process:



We see that the maximum sampling rate f_s that gives a spectrally flat sequence is obtained when :

$$W = f_s - W \Rightarrow f_s = 2W \Rightarrow T = \frac{1}{2W}$$

(c) The triangular-shaped spectrum $\Phi(f) = 1 - \frac{|f|}{W}$, $|f| \leq W$ may be obtained by convolving the rectangular-shaped spectrum $\Phi_1(f) = 1/\sqrt{W}$, $|f| \leq W/2$. Hence, $\phi(\tau) = \phi_1^2(\tau) =$

$\frac{1}{W}\left(\frac{\sin \pi W\tau}{\pi\tau}\right)^2$.Therefore, sampling X(t) at a rate $\frac{1}{T}=W$ samples/sec produces a white sequence with autocorrelation function :

$$\phi_d(k)=\frac{1}{W}\left(\frac{\sin \pi WkT}{\pi kT}\right)^2=W\left(\frac{\sin \pi k}{\pi k}\right)^2=\left\{\begin{array}{cc} W, & \text{k}=0 \\ 0, & \text{otherwise}\end{array}\right\}$$

Problem 2.23 :

Let's denote : $y(t)=f_k(t)f_j(t)$.Then :

$$\int_{-\infty}^{\infty} f_k(t)f_j(t)dt=\int_{-\infty}^{\infty} y(t)dt=Y(f)|_{f=0}$$

where $Y(f)$ is the Fourier transform of $y(t)$. Since : $y(t)=f_k(t)f_j(t)\longleftrightarrow Y(f)=F_k(f)*F_j(f)$. But :

$$F_k(f)=\int_{-\infty}^{\infty} f_k(t)e^{-j2\pi ft}dt=\frac{1}{2W}e^{-j2\pi fk/2W}$$

Then :

$$Y(f)=F_k(f)*F_j(f)=\int_{-\infty}^{\infty} F_k(a)*F_j(f-a)da$$

and at $f=0$:

$$\begin{array}{rcl} Y(f)|_{f=0} & = & \int_{-\infty}^{\infty} F_k(a)*F_j(-a)da \\ & = & \left(\frac{1}{2W}\right)^2 \int_{-\infty}^{\infty} e^{-j2\pi a(k-j)/2W}da \\ & = & \left\{\begin{array}{cc} 1/2W, & \text{k}=\text{j} \\ 0, & \text{k}\neq\text{j}\end{array}\right\}\end{array}$$

Problem 2.24 :

$$B_{eq}=\frac{1}{G}\int_0^{\infty}|H(f)|^2df$$

For the filter shown in Fig. P2-12 we have $G=1$ and

$$B_{eq}=\int_0^{\infty}|H(f)|^2df=B$$

For the lowpass filter shown in Fig. P2-16 we have

$$H(f)=\frac{1}{1+j2\pi fRC}\Rightarrow|H(f)|^2=\frac{1}{1+(2\pi fRC)^2}$$

So $G = 1$ and

$$\begin{aligned} B_{eq} &= \textstyle\int_0^\infty |H(f)|^2 df \\ &= \textstyle\frac{1}{2}\int_{-\infty}^\infty |H(f)|^2 df \\ &= \textstyle\frac{1}{4RC} \end{aligned}$$

where the last integral is evaluated in the same way as in problem P-2.9 .

CHAPTER 3

Problem 3.1 :

$$I(B_j;A_i) = \log_2 \frac{P(B_j|A_i)}{P(B_j)} = \log_2 \frac{P(B_j,A_i)}{P(B_j)P(A_i)}$$

Also :

$$P(B_j) = \sum_{i=1}^{4} P(B_j,A_i) = \left\{ \begin{array}{ll} 0.31, & \text{j} = 1 \\ 0.27, & \text{j} = 2 \\ 0.42, & \text{j} = 3 \end{array} \right\}$$

$$P(A_i) = \sum_{j=1}^{3} P(B_j,A_i) = \left\{ \begin{array}{ll} 0.31, & \text{i} = 1 \\ 0.17, & \text{i} = 2 \\ 0.31, & \text{i} = 3 \\ 0.21, & \text{i} = 4 \end{array} \right\}$$

Hence :

$$I(B_1;A_1) = \log_2 \frac{0.10}{(0.31)(0.31)} = +0.057 \text{ bits}$$

$$I(B_1;A_2) = \log_2 \frac{0.05}{(0.31)(0.17)} = -0.076 \text{ bits}$$

$$I(B_1;A_3) = \log_2 \frac{0.05}{(0.31)(0.31)} = -0.943 \text{ bits}$$

$$I(B_1;A_4) = \log_2 \frac{0.11}{(0.31)(0.21)} = +0.757 \text{ bits}$$

$$I(B_2;A_1) = \log_2 \frac{0.08}{(0.27)(0.31)} = -0.065 \text{ bits}$$

$$I(B_2;A_2) = \log_2 \frac{0.03}{(0.27)(0.17)} = -0.614 \text{ bits}$$

$$I(B_2;A_3) = \log_2 \frac{0.12}{(0.27)(0.31)} = +0.520 \text{ bits}$$

$$I(B_2;A_4) = \log_2 \frac{0.04}{(0.27)(0.21)} = -0.503 \text{ bits}$$

$$I(B_3;A_1) = \log_2 \frac{0.13}{(0.42)(0.31)} = -0.002 \text{ bits}$$

$$I(B_3;A_2) = \log_2 \frac{0.09}{(0.42)(0.17)} = +0.334 \text{ bits}$$

$$I(B_3;A_3) = \log_2 \frac{0.14}{(0.42)(0.31)} = +0.105 \text{ bits}$$
$$I(B_3;A_4) = \log_2 \frac{0.06}{(0.42)(0.21)} = -0.556 \text{ bits}$$

(b) The average mutual information will be :

$$I(B;A) = \sum_{j=1}^{3}\sum_{i=1}^{4} P(Ai, B_j) I(B_j;A_i) = 0.677 \text{ bits}$$

Problem 3.2 :

$$\begin{aligned} H(B) &= -\textstyle\sum_{j=1}^{3} P(B_j)\log_2 P(B_j) \\ &= -\left[0.31\log_2 0.31 + 0.27\log_2 0.27 + 0.42\log_2 0.42\right] \\ &= 1.56 \text{ bits/letter} \end{aligned}$$

Problem 3.3 :

Let $f(u) = u - 1 - \ln u$. The first and second derivatives of $f(u)$ are

$$\frac{df}{du} = 1 - \frac{1}{u}$$

and

$$\frac{d^2 f}{du^2} = \frac{1}{u^2} > 0, \ \forall u > 0$$

Hence this function achieves its minimum at $\frac{df}{du} = 0 \Rightarrow u = 1$. The minimum value is $f(u = 1) = 0$ so $\ln u = u - 1$, at $u = 1$. For all other values of $u : 0 < u < \infty, \ u \neq 1$, we have $f(u) > 0 \Rightarrow u - 1 > \ln u$.

Problem 3.4 :

We will show that $-I(X;Y) \leq 0$

$$\begin{aligned} -I(X;Y) &= -\sum_i \sum_j P(x_i, y_j) \log_2 \frac{P(x_i,y_j)}{P(x_i)P(y_j)} \\ &= \frac{1}{\ln 2} \sum_i \sum_j P(x_i, y_j) \ln \frac{P(x_i)P(y_j)}{P(x_i,y_j)} \end{aligned}$$

We use the inequality $\ln u \leq u - 1$. We need only consider those terms for which $P(x_i, y_j) > 0$; then, applying the inequality to each term in $I(X;Y)$:

$$\begin{aligned} -I(X;Y) &\leq \frac{1}{\ln 2} \sum_i \sum_j P(x_i, y_j) \left[\frac{P(x_i)P(y_j)}{P(x_i,y_j)} - 1\right] \\ &= \frac{1}{\ln 2} \sum_i \sum_j \left[P(x_i)P(y_j) - P(x_i, y_j)\right] \leq 0 \end{aligned}$$

The first inequality becomes equality if and only if

$$\frac{P(x_i)P(y_j)}{P(x_i, y_j)} = 1 \Longleftrightarrow P(x_i)P(y_j) = P(x_i, y_j)$$

when $P(x_i, y_j) > 0$. Also, since the summations

$$\sum_i \sum_j \left[P(x_i)P(y_j) - P(x_i, y_j)\right]$$

contain only the terms for which $P(x_i, y_j) > 0$, this term equals zero if and only if $P(X_i)P(Y_j) = 0$, when $P(x_i, y_j) = 0$. Therefore, both inequalitites become equalities and hence, $I(X;Y) = 0$ if and only if X and Y are statistically independent.

Problem 3.5 :

We shall prove that $H(X) - \log n \leq 0$:

$$\begin{aligned} H(X) - \log n &= \sum_{i=1}^n p_i \log \frac{1}{p_i} - \log n \\ &= \sum_{i=1}^n p_i \log \frac{1}{p_i} - \sum_{i=1}^n p_i \log n \\ &= \sum_{i=1}^n p_i \log \frac{1}{np_i} \\ &= \frac{1}{\ln 2} \sum_{i=1}^n p_i \ln \frac{1}{np_i} \\ &\leq \frac{1}{\ln 2} \sum_{i=1}^n p_i \left(\frac{1}{np_i} - 1 \right) \\ &= 0 \end{aligned}$$

Hence, $H(X) \leq \log n$. Also, if $p_i = 1/n \quad \forall\, i \Rightarrow H(X) = \log n$.

Problem 3.6 :

By definition, the differential entropy is

$$H(X) = -\int_{-\infty}^{\infty} p(x) \log p(x) dx$$

For the uniformly distributed random variable :

$$H(X) = -\int_0^a \frac{1}{a} \log \frac{1}{a} dx = \log a$$

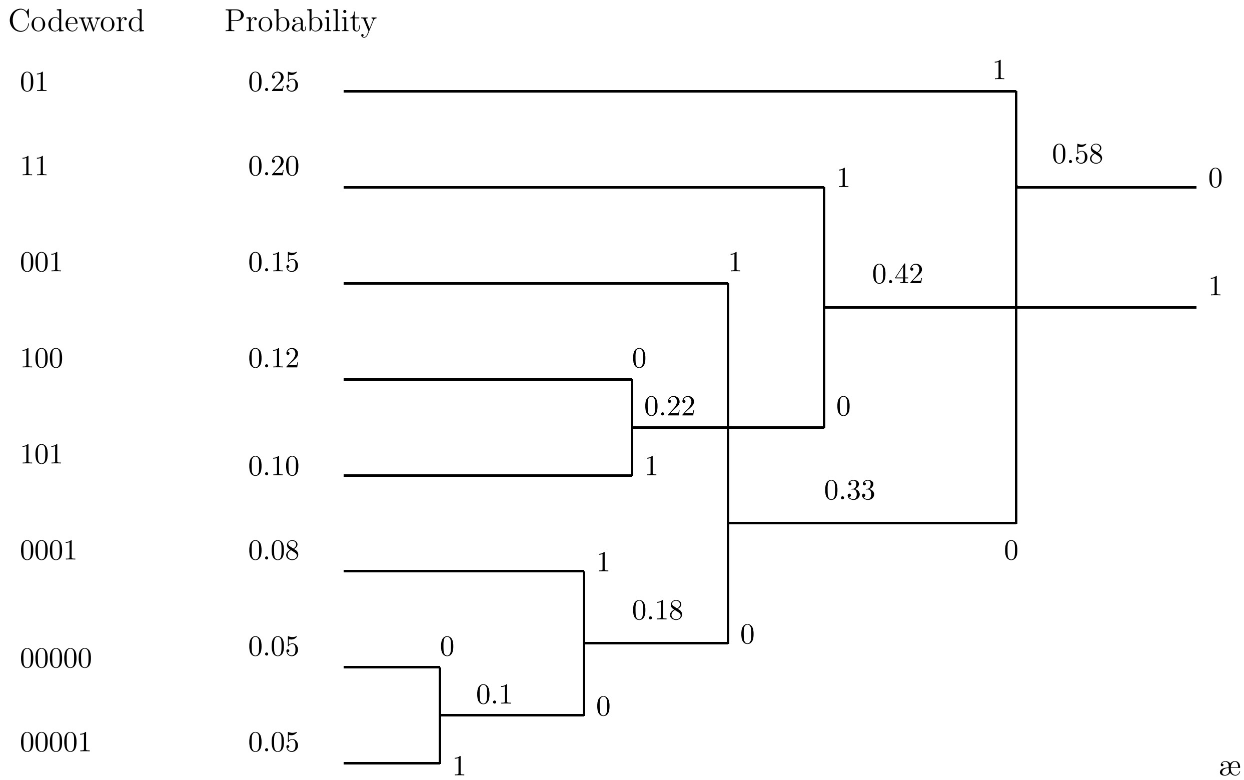
(a) For $a = 1, \quad H(X) = 0$

(b) For $a = 4, \quad H(X) = \log 4 = 2 \log 2$

(c) For $a = 1/4, \quad H(X) = \log \frac{1}{4} = -2 \log 2$

Problem 3.7 :

(a) The following figure depicts the design of a ternary Huffman code (we follow the convention that the lower-probability branch is assigned a 1) :



(b) The average number of binary digits per source letter is :

$$\bar{R} = \sum_i P(x_i) n_i = 2(0.45) + 3(0.37) + 4(0.08) + 5(0.1) = 2.83 \text{ bits/letter}$$

(c) The entropy of the source is :

$$H(X) = -\sum_i P(x_i) log P(x_i) = 2.80 \text{ bits/letter}$$

As it is expected the entropy of the source is less than the average length of each codeword.

Problem 3.8 :

The source entropy is :

$$H(X) = \sum_{i=1}^{5} p_i \log \frac{1}{p_i} = \log 5 = 2.32 \text{ bits/letter}$$

(a) When we encode one letter at a time we require $\bar{R} = 3$ bits/letter . Hence, the efficiency is $2.32/3 = 0.77$ (77%).

(b) If we encode two letters at a time, we have 25 possible sequences. Hence, we need 5 bits per 2-letter symbol, or $\bar{R} = 2.5$ bits/letter ; the efficiency is $2.32/2.5 = 0.93$.

(c) In the case of encoding three letters at a time we have 125 possible sequences. Hence we need 7 bits per 3-letter symbol, so $\bar{R} = 7/3$ bits/letter; the efficiency is $2.32/(7/3) = 0.994$.

Problem 3.9 :

(a)

$$\begin{aligned} I(x_i; y_j) &= \log \tfrac{P(x_i|y_j)}{P(x_i)} \\ &= \log \tfrac{P(x_i,y_j)}{P(x_i)P(y_j)} \\ &= \log \tfrac{P(y_j|x_i)}{P(y_j)} \\ &= \log \tfrac{1}{P(y_j)} - \log \tfrac{1}{P(y_j|x_i)} \\ &= I(y_j) - I(y_j|x_i) \end{aligned}$$

(b)

$$\begin{aligned} I(x_i; y_j) &= \log \tfrac{P(x_i|y_j)}{P(x_i)} \\ &= \log \tfrac{P(x_i,y_j)}{P(x_i)P(y_j)} \\ &= \log \tfrac{1}{P(x_i)} + \log \tfrac{1}{P(y_j)} - \log \tfrac{1}{P(x_i,y_j)} \\ &= I(x_i) + I(y_j) - I(x_i, y_j) \end{aligned}$$

Problem 3.10 :

(a)

$$\begin{aligned} H(X) &= -\sum_{k=1}^{\infty} p(1-p)^{k-1} \log_2(p(1-p)^{k-1}) \\ &= -p\log_2(p)\sum_{k=1}^{\infty}(1-p)^{k-1} - p\log_2(1-p)\sum_{k=1}^{\infty}(k-1)(1-p)^{k-1} \\ &= -p\log_2(p)\frac{1}{1-(1-p)} - p\log_2(1-p)\frac{1-p}{(1-(1-p))^2} \\ &= -\log_2(p) - \frac{1-p}{p}\log_2(1-p) \end{aligned}$$

(b) Clearly $P(X = k|X > K) = 0$ for $k \leq K$. If $k > K$, then

$$P(X = k|X > K) = \frac{P(X = k, X > K)}{P(X > K)} = \frac{p(1-p)^{k-1}}{P(X > K)}$$

But,

$$\begin{aligned} P(X > K) &= \sum_{k=K+1}^{\infty} p(1-p)^{k-1} = p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} - \sum_{k=1}^{K}(1-p)^{k-1}\right) \\ &= p\left(\frac{1}{1-(1-p)} - \frac{1-(1-p)^K}{1-(1-p)}\right) = (1-p)^K \end{aligned}$$

so that

$$P(X = k|X > K) = \frac{p(1-p)^{k-1}}{(1-p)^K}$$

If we let $k = K + l$ with $l = 1, 2, \ldots$, then

$$P(X = k|X > K) = \frac{p(1-p)^K(1-p)^{l-1}}{(1-p)^K} = p(1-p)^{l-1}$$

that is $P(X = k|X > K)$ is the geometrically distributed. Hence, using the results of the first part we obtain

$$\begin{aligned} H(X|X > K) &= -\sum_{l=1}^{\infty} p(1-p)^{l-1}\log_2(p(1-p)^{l-1}) \\ &= -\log_2(p) - \frac{1-p}{p}\log_2(1-p) \end{aligned}$$

Problem 3.11 :

(a) The marginal distribution $P(x)$ is given by $P(x) = \sum_y P(x, y)$. Hence,

$$\begin{aligned} H(X) &= -\sum_x P(x)\log P(x) = -\sum_x\sum_y P(x, y)\log P(x) \\ &= -\sum_{x,y} P(x, y)\log P(x) \end{aligned}$$

Similarly it is proved that $H(Y) = -\sum_{x,y} P(x, y)\log P(y)$.

(b) Using the inequality $\ln w \leq w - 1$ with $w = \frac{P(x)P(y)}{P(x,y)}$, we obtain

$$\ln\frac{P(x)P(y)}{P(x, y)} \leq \frac{P(x)P(y)}{P(x, y)} - 1$$

Multiplying the previous by $P(x,y)$ and adding over x, y, we obtain

$$\sum_{x,y} P(x,y) \ln P(x)P(y) - \sum_{x,y} P(x,y) \ln P(x,y) \leq \sum_{x,y} P(x)P(y) - \sum_{x,y} P(x,y) = 0$$

Hence,

$$\begin{aligned} H(X,Y) &\leq -\sum_{x,y} P(x,y) \ln P(x)P(y) = -\sum_{x,y} P(x,y)(\ln P(x) + \ln P(y)) \\ &= -\sum_{x,y} P(x,y) \ln P(x) - \sum_{x,y} P(x,y) \ln P(y) = H(X) + H(Y) \end{aligned}$$

Equality holds when $\frac{P(x)P(y)}{P(x,y)} = 1$, i.e when X, Y are independent.

(c)

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

Also, from part (b), $H(X,Y) \leq H(X) + H(Y)$. Combining the two relations, we obtain

$$H(Y) + H(X|Y) \leq H(X) + H(Y) \Longrightarrow H(X|Y) \leq H(X)$$

Suppose now that the previous relation holds with equality. Then,

$$-\sum_x P(x) \log P(x|y) = -\sum_x P(x) \log P(x) \Rightarrow \sum_x P(x) \log(\frac{P(x)}{P(x|y)}) = 0$$

However, $P(x)$ is always greater or equal to $P(x|y)$, so that $\log(P(x)/P(x|y))$ is non-negative. Since $P(x) > 0$, the above equality holds if and only if $\log(P(x)/P(x|y)) = 0$ or equivalently if and only if $P(x)/P(x|y) = 1$. This implies that $P(x|y) = P(x)$ meaning that X and Y are independent.

Problem 3.12 :

The marginal probabilities are given by

$$\begin{aligned} P(X=0) &= \sum_k P(X=0,Y=k) = P(X=0,Y=0) + P(X=0,Y=1) = \frac{2}{3} \\ P(X=1) &= \sum_k P(X=1,Y=k) = P(X=1,Y=1) = \frac{1}{3} \\ P(Y=0) &= \sum_k P(X=k,Y=0) = P(X=0,Y=0) = \frac{1}{3} \\ P(Y=1) &= \sum_k P(X=k,Y=1) = P(X=0,Y=1) + P(X=1,Y=1) = \frac{2}{3} \end{aligned}$$

Hence,

$$\begin{aligned}
H(X) &= -\sum_{i=0}^{1} P_i \log_2 P_i = -(\frac{1}{3}\log_2 \frac{1}{3} + \frac{1}{3}\log_2 \frac{1}{3}) = .9183 \\
H(X) &= -\sum_{i=0}^{1} P_i \log_2 P_i = -(\frac{1}{3}\log_2 \frac{1}{3} + \frac{1}{3}\log_2 \frac{1}{3}) = .9183 \\
H(X,Y) &= -\sum_{i=0}^{2} \frac{1}{3}\log_2 \frac{1}{3} = 1.5850 \\
H(X|Y) &= H(X,Y) - H(Y) = 1.5850 - 0.9183 = 0.6667 \\
H(Y|X) &= H(X,Y) - H(X) = 1.5850 - 0.9183 = 0.6667
\end{aligned}$$

Problem 3.13 :

$$\begin{aligned}
H &= \lim_{n\to\infty} H(X_n|X_1,\ldots,X_{n-1}) \\
&= \lim_{n\to\infty} \left[-\sum_{x_1,\ldots,x_n} P(x_1,\ldots,x_n) \log_2 P(x_n|x_1,\ldots,x_{n-1}) \right] \\
&= \lim_{n\to\infty} \left[-\sum_{x_1,\ldots,x_n} P(x_1,\ldots,x_n) \log_2 P(x_n|x_{n-1}) \right] \\
&= \lim_{n\to\infty} \left[-\sum_{x_n,x_{n-1}} P(x_n,x_{n-1}) \log_2 P(x_n|x_{n-1}) \right] \\
&= \lim_{n\to\infty} H(X_n|X_{n-1})
\end{aligned}$$

However, for a stationary process $P(x_n, x_{n-1})$ and $P(x_n|x_{n-1})$ are independent of n, so that

$$H = \lim_{n\to\infty} H(X_n|X_{n-1}) = H(X_n|X_{n-1})$$

Problem 3.14 :

$$\begin{aligned}
H(X,Y) &= H(X,g(X)) = H(X) + H(g(X)|X) \\
&= H(g(X)) + H(X|g(X))
\end{aligned}$$

But, $H(g(X)|X) = 0$, since $g(\cdot)$ is deterministic. Therefore,

$$H(X) = H(g(X)) + H(X|g(X))$$

Since each term in the previous equation is non-negative we obtain

$$H(X) \geq H(g(X))$$

Equality holds when $H(X|g(X)) = 0$. This means that the values $g(X)$ uniquely determine X, or that $g(\cdot)$ is a one to one mapping.

Problem 3.15 :

$$\begin{aligned}
I(X;Y) &= \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log\frac{P(x_i,y_j)}{P(x_i)P(y_j)} \\
&= \left\{ \begin{array}{l} \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log P(x_i, y_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log P(x_i) \\ -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log P(y_j) \end{array} \right\} \\
&= \left\{ \begin{array}{l} \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j)\log P(x_i, y_j) - \sum_{i=1}^{n} P(x_i)\log P(x_i) \\ -\sum_{j=1}^{m} P(y_j)\log P(y_j) \end{array} \right\} \\
&= -H(XY) + H(X) + H(Y)
\end{aligned}$$

Problem 3.16 :

$$H(X_1X_2...X_n) = -\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}...\sum_{j_n=1}^{m_n} P(x_1, x_2, ..., x_n)\log P(x_1, x_2, ..., x_n)$$

Since the $\{x_i\}$ are statistically independent :

$$P(x_1, x_2, ..., x_n) = P(x_1)P(x_2)...P(x_n)$$

and

$$\sum_{j_2=1}^{m_2}...\sum_{j_n=1}^{m_n} P(x_1)P(x_2)...P(x_n) = P(x_1)$$

(similarly for the other x_i). Then :

$$\begin{aligned}
H(X_1X_2...X_n) &= -\sum_{j_1=1}^{m_1}\sum_{j_2=1}^{m_2}...\sum_{j_n=1}^{m_n} P(x_1)P(x_2)...P(x_n)\log P(x_1)P(x_2)...P(x_n) \\
&= -\sum_{j_1=1}^{m_1} P(x_1)\log P(x_1) - \sum_{j_2=1}^{m_2} P(x_2)\log P(x_2)... - \sum_{j_n=1}^{m_n} P(x_n)\log P(x_n) \\
&= \sum_{i=1}^{n} H(X_i)
\end{aligned}$$

Problem 3.17 :

We consider an $n-$input, $n-$output channel. Since it is noiseless :

$$P(y_j|x_i) = \left\{ \begin{array}{ll} 0, & \mathrm{i} \neq \mathrm{j} \\ 1, & \mathrm{i} = \mathrm{j} \end{array} \right\}$$

Hence :

$$\begin{array}{rcl} H(X|Y) & = & \sum_{i=1}^n \sum_{j=1}^n P(x_i, y_j) \log P(x_i|y_j) \\ & = & \sum_{i=1}^n \sum_{j=1}^n P(y_j|x_i) p(x_i) \log P(x_i|y_j) \end{array}$$

But it is also true that :

$$P(x_i|y_j) = \left\{ \begin{array}{ll} 0, & \mathrm{i} \neq \mathrm{j} \\ 1, & \mathrm{i} = \mathrm{j} \end{array} \right\}$$

Hence :

$$H(X|Y) = -\sum_{i=1}^n P(x_i) \log 1 = 0$$

Problem 3.18 :

The conditional mutual information between x_3 and x_2 given x_1 is defined as :

$$I(x_3; x_2|x_1) = \log \frac{P(x_3, x_2|x_1)}{P(x_3|x_1)P(x_2|x_1)} = \log \frac{P(x_3|x_2 x_1)}{P(x_3|x_1)}$$

Hence :

$$I(x_3; x_2|x_1) = I(x_3|x_1) - I(x_3|x_2 x_1)$$

and

$$\begin{array}{rcl} I(X_3; X_2|X_1) & = & \sum_{x1} \sum_{x2} \sum_{x3} P(x_1, x_2, x_3) \log \frac{P(x_3|x_2 x_1)}{P(x_3|x_1)} \\ & = & \left\{ \begin{array}{l} -\sum_{x1} \sum_{x2} \sum_{x3} P(x_1, x_2, x_3) \log P(x_3|x_1) \\ +\sum_{x1} \sum_{x2} \sum_{x3} P(x_1, x_2, x_3) \log P(x_3|x_2 x_1) \end{array} \right\} \\ & = & H(X_3|X_1) - H(X_3|X_2 X_1) \end{array}$$

Since $I(X_3; X_2|X_1) \geq 0$, it follows that :

$$H(X_3|X_1) \geq H(X_3|X_2 X_1)$$

Problem 3.19 :

Assume that $a > 0$. Then we know that in the linear transformation $Y = aX + b$:

$$p_Y(y) = \frac{1}{a} p_X(\frac{y-b}{a})$$

Hence :

$$\begin{aligned} H(Y) &= -\int_{-\infty}^{\infty} p_Y(y) \log p_Y(y) dy \\ &= -\int_{-\infty}^{\infty} \frac{1}{a} p_X(\frac{y-b}{a}) \log \frac{1}{a} p_X(\frac{y-b}{a}) dy \end{aligned}$$

Let $u = \frac{y-b}{a}$. Then $dy = adu$, and :

$$\begin{aligned} H(Y) &= -\int_{-\infty}^{\infty} \frac{1}{a} p_X(u) \left[\log p_X(u) - \log a\right] adu \\ &= -\int_{-\infty}^{\infty} p_X(u) \log p_X(u) du + \int_{-\infty}^{\infty} p_X(u) \log a du \\ &= H(X) + \log a \end{aligned}$$

In a similar way, we can prove that for $a < 0$:

$$H(Y) = -H(X) - \log a$$

Problem 3.20 :

The linear transformation produces the symbols :

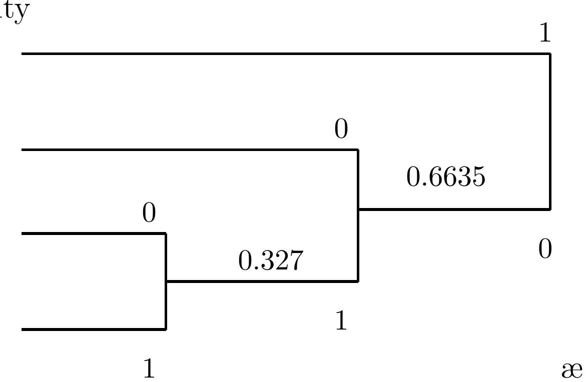
$$y_i = ax_i + b, \quad i = 1, 2, 3$$

with corresponding probabilities $p_1 = 0.45,\ p_2 = 0.35,\ p_3 = 0.20$. since the $\{y_i\}$ have the same probability distribution as the $\{x_i\}$, it follows that : $H(Y) = H(X)$. Hence, the entropy of a DMS is not affected by the linear transformation.

Problem 3.21 :

(a) The following figure depicts the design of the Huffman code, when encoding a single level at a time :

Codeword	Level	Probability
1	a_1	0.3365
00	a_2	0.3365
010	a_3	0.1635
011	a_4	0.1635



The average number of binary digits per source level is :

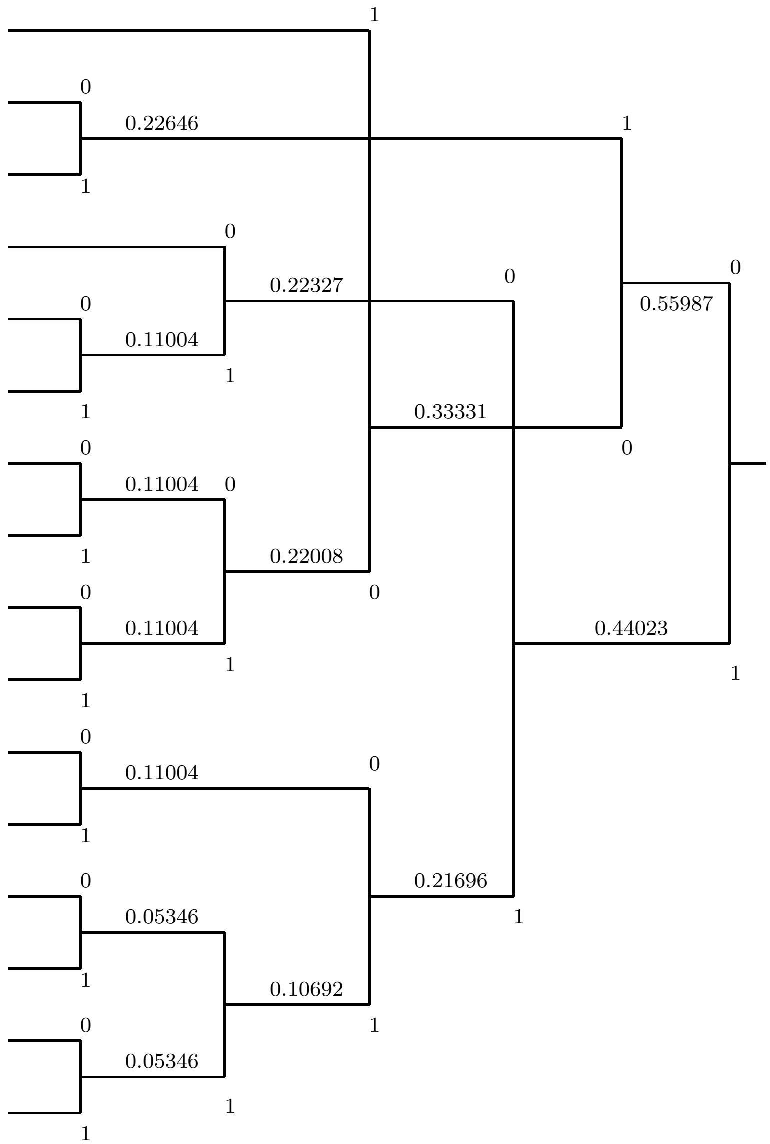
$$\bar{R} = \sum_i P(a_i)n_i = 1.9905 \text{ bits/level}$$

The entropy of the source is :

$$H(X) = -\sum_i P(a_i) log P(a_i) = 1.9118 \text{ bits/level}$$

(b) Encoding two levels at a time :

Codeword	Levels	Probability
001	a_1a_1	0.11323
010	a_1a_2	0.11323
011	a_2a_1	0.11323
100	a_2a_2	0.11323
1011	a_1a_3	0.05502
1010	a_1a_4	0.05502
00000	a_2a_3	0.05502
00001	a_2a_4	0.05502
00010	a_3a_1	0.05502
00011	a_3a_2	0.05502
1100	a_4a_1	0.05502
1101	a_4a_2	0.05502
11100	a_3a_3	0.02673
11101	a_3a_4	0.02673
11110	a_4a_3	0.02673
11111	a_4a_4	0.02673



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The average number of binary digits per level pair is $\bar{R}_2 = \sum_k P(\mathbf{a_k})n_k = 3.874$ bits/pair resulting in an average number :

$$\bar{R} = 1.937 \text{ bits/level}$$

(c)

$$H(X) \leq \frac{\bar{R}_J}{J} < H(X) + \frac{1}{J}$$

As $J \rightarrow \infty$, $\frac{\bar{R}_J}{J} \rightarrow H(X) = 1.9118$ bits/level.

Problem 3.22 :

First, we need the state probabilities $P(x_i),\ i = 1, 2$. For stationary Markov processes, these can be found, in general, by the solution of the system :

$$P\Pi = P, \quad \sum_i P_i = 1$$

where P is the state probability vector and Π is the transition matrix : $\Pi[ij] = P(x_j|x_i)$. However, in the case of a two-state Markov source, we can find $P(x_i)$ in a simpler way by noting that the probability of a transition from state 1 to state 2 equals the probability of a transition from state 2 to state 1(so that the probability of each state will remain the same). Hence :

$$P(x_1|x_2)P(x_2) = P(x_2|x_1)P(x_1) \Rightarrow 0.3P(x_2) = 0.2P(x_1) \Rightarrow P(x_1) = 0.6,\ P(x_2) = 0.4$$

Then :

$$\begin{aligned} H(X) &= \left\{ \begin{array}{l} P(x_1)\left[-P(x_1|x_1)\log P(x_1|x_1) - P(x_2|x_1)\log P(x_2|x_1)\right] + \\ P(x_2)\left[-P(x_1|x_2)\log P(x_1|x_2) - P(x_2|x_2)\log P(x_2|x_2)\right] \end{array} \right\} \\ &= 0.6\left[-0.8\log 0.8 - 0.2\log 0.2\right] + 0.4\left[-0.3\log 0.3 - 0.7\log 0.7\right] \\ &= 0.7857 \text{ bits/letter} \end{aligned}$$

If the source is a binary DMS with output letter probabilities $P(x_1) = 0.6,\ P(x_2) = 0.4$, its entropy will be :

$$H_{DMS}(X) = -0.6\log 0.6 - 0.4\log 0.4 = 0.971 \text{ bits/letter}$$

We see that the entropy of the Markov source is smaller, since the memory inherent in it reduces the information content of each output.

Problem 3.23 :

(a)

$$\begin{aligned} H(X) &= -(.05\log_2 .05 + .1\log_2 .1 + .1\log_2 .1 + .15\log_2 .15 \\ &\quad +.05\log_2 .05 + .25\log_2 .25 + .3\log_2 .3) = 2.5282 \end{aligned}$$

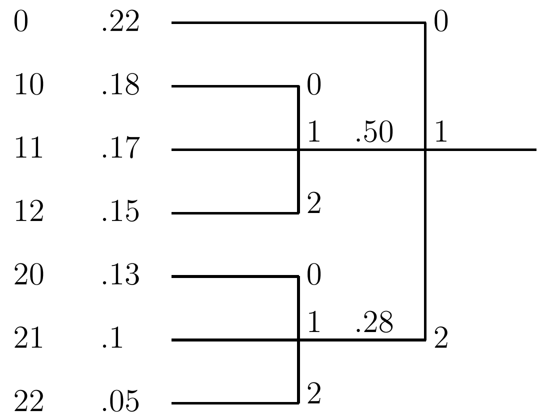
(b) After quantization, the new alphabet is $B = \{-4, 0, 4\}$ and the corresponding symbol probabilities are given by

$$\begin{aligned} P(-4) &= P(-5) + P(-3) = .05 + .1 = .15 \\ P(0) &= P(-1) + P(0) + P(1) = .1 + .15 + .05 = .3 \\ P(4) &= P(3) + P(5) = .25 + .3 = .55 \end{aligned}$$

Hence, $H(Q(X)) = 1.4060$. As it is observed quantization decreases the entropy of the source.

Problem 3.24 :

The following figure depicts the design of a ternary Huffman code.



The average codeword length is

$$\begin{aligned}\bar{R}(X) &= \sum_x P(x)n_x = .22 + 2(.18 + .17 + .15 + .13 + .10 + .05) \\ &= 1.78 \quad \text{(ternary symbols/output)}\end{aligned}$$

For a fair comparison of the average codeword length with the entropy of the source, we compute the latter with logarithms in base 3. Hence,

$$H(X) = -\sum_x P(x) \log_3 P(x) = 1.7047$$

As it is expected $H(X) \leq \bar{R}(X)$.

Problem 3.25 :

Parsing the sequence by the rules of the Lempel-Ziv coding scheme we obtain the phrases
0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001,
000000, 11, 01, 0000000, 110, ...
The number of the phrases is 18. For each phrase we need 5 bits plus an extra bit to represent the new source output.

Dictionary Location	Dictionary Contents	Codeword
1 00001	0	00000 0
2 00010	00	00001 0
3 00011	1	00000 1
4 00100	001	00010 1
5 00101	000	00010 0
6 00110	0001	00101 1
7 00111	10	00011 0
8 01000	00010	00110 0
9 01001	0000	00101 0
10 01010	0010	00100 0
11 01011	00000	01001 0
12 01100	101	00111 1
13 01101	00001	01001 1
14 01110	000000	01011 0
15 01111	11	00011 1
16 10000	01	00001 1
17 10001	0000000	01110 0
18 10010	110	01111 0

Problem 3.26 :

(a)

$$\begin{aligned} H(X) &= -\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}} \ln(\frac{1}{\lambda}e^{-\frac{x}{\lambda}})dx \\ &= -\ln(\frac{1}{\lambda})\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx + \int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}\frac{x}{\lambda}dx \\ &= \ln\lambda + \frac{1}{\lambda}\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}xdx \\ &= \ln\lambda + \frac{1}{\lambda}\lambda = 1 + \ln\lambda \end{aligned}$$

where we have used the fact $\int_0^\infty \frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx = 1$ and $E[X] = \int_0^\infty x\frac{1}{\lambda}e^{-\frac{x}{\lambda}}dx = \lambda$.

(b)

$$\begin{aligned} H(X) &= -\int_{-\infty}^\infty \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}} \ln(\frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}})dx \\ &= -\ln(\frac{1}{2\lambda})\int_{-\infty}^\infty \frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}dx + \frac{1}{\lambda}\int_{-\infty}^\infty |x|\frac{1}{2\lambda}e^{-\frac{|x|}{\lambda}}dx \end{aligned}$$

$$
\begin{aligned}
&= \ln(2\lambda) + \frac{1}{\lambda}\left[\int_{-\infty}^{0} -x\frac{1}{2\lambda}e^{\frac{x}{\lambda}}dx + \int_{0}^{\infty} x\frac{1}{2\lambda}e^{-\frac{x}{\lambda}}dx\right] \\
&= \ln(2\lambda) + \frac{1}{2\lambda}\lambda + \frac{1}{2\lambda}\lambda = 1 + \ln(2\lambda)
\end{aligned}
$$

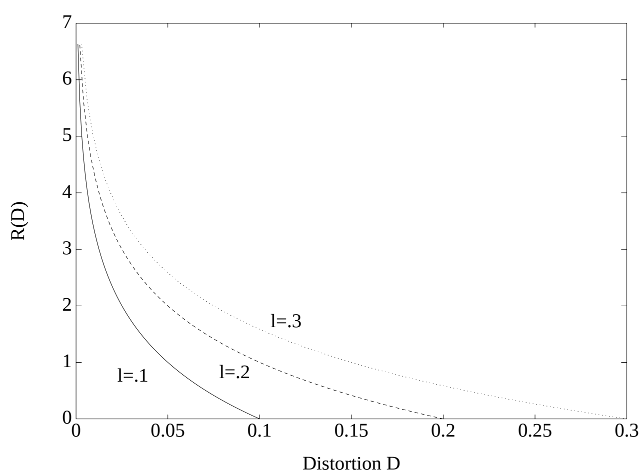
(c)

$$
\begin{aligned}
H(X) &= -\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2}\ln\left(\frac{x+\lambda}{\lambda^2}\right)dx - \int_{0}^{\lambda} \frac{-x+\lambda}{\lambda^2}\ln\left(\frac{-x+\lambda}{\lambda^2}\right)dx \\
&= -\ln\left(\frac{1}{\lambda^2}\right)\left[\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2}dx + \int_{0}^{\lambda} \frac{-x+\lambda}{\lambda^2}dx\right] \\
&\quad -\int_{-\lambda}^{0} \frac{x+\lambda}{\lambda^2}\ln(x+\lambda)dx - \int_{0}^{\lambda} \frac{-x+\lambda}{\lambda^2}\ln(-x+\lambda)dx \\
&= \ln(\lambda^2) - \frac{2}{\lambda^2}\int_{0}^{\lambda} z\ln z dz \\
&= \ln(\lambda^2) - \frac{2}{\lambda^2}\left[\frac{z^2\ln z}{2} - \frac{z^2}{4}\right]_0^{\lambda} \\
&= \ln(\lambda^2) - \ln(\lambda) + \frac{1}{2}
\end{aligned}
$$

Problem 3.27 :

(a) Since $R(D) = \log\frac{\lambda}{D}$ and $D = \frac{\lambda}{2}$, we obtain $R(D) = \log(\frac{\lambda}{\lambda/2}) = \log(2) = 1$ bit/sample.

(b) The following figure depicts $R(D)$ for $\lambda = 0.1$, .2 and .3. As it is observed from the figure, an increase of the parameter λ increases the required rate for a given distortion.



Problem 3.28 :

(a) For a Gaussian random variable of zero mean and variance σ^2 the rate-distortion function is given by $R(D) = \frac{1}{2}\log_2\frac{\sigma^2}{D}$. Hence, the upper bound is satisfied with equality. For the lower bound recall that $H(X) = \frac{1}{2}\log_2(2\pi e\sigma^2)$. Thus,

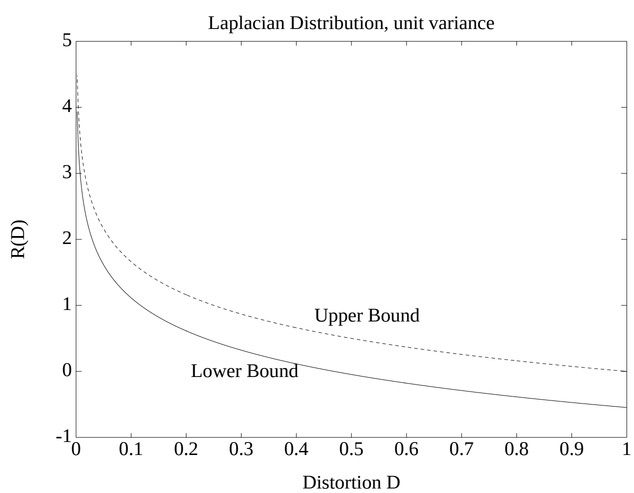
$$\begin{aligned} H(X) - \frac{1}{2}\log_2(2\pi eD) &= \frac{1}{2}\log_2(2\pi e\sigma^2) - \frac{1}{2}\log_2(2\pi eD) \\ &= \frac{1}{2}\log_2\left(\frac{2\pi e\sigma^2}{2\pi eD}\right) = R(D) \end{aligned}$$

As it is observed the upper and the lower bounds coincide.

(b) The differential entropy of a Laplacian source with parameter λ is $H(X) = 1 + \ln(2\lambda)$. The variance of the Laplacian distribution is

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \frac{1}{2\lambda} e^{-\frac{|x|}{\lambda}} dx = 2\lambda^2$$

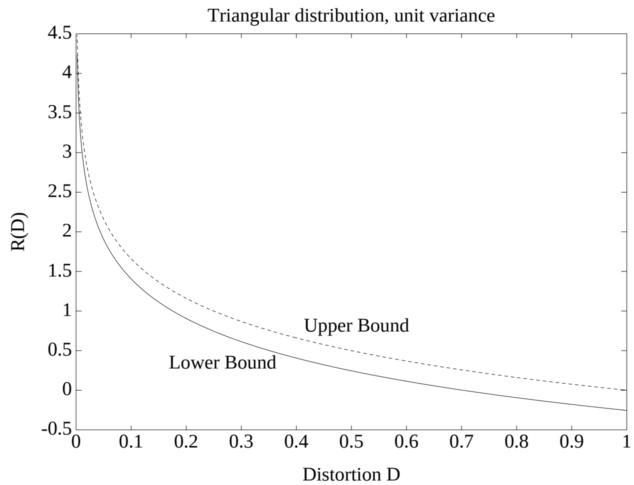
Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{1/2}$ and $H(X) = 1+\ln(2\lambda) = 1+\ln(\sqrt{2}) = 1.3466$ nats/symbol = 1.5 bits/symbol. A plot of the lower and upper bound of $R(D)$ is given in the next figure.



(c) The variance of the triangular distribution is given by

$$\begin{aligned} \sigma^2 &= \int_{-\lambda}^{0}\left(\frac{x+\lambda}{\lambda^2}\right)x^2dx + \int_0^{\lambda}\left(\frac{-x+\lambda}{\lambda^2}\right)x^2dx \\ &= \frac{1}{\lambda^2}\left(\frac{1}{4}x^4 + \frac{\lambda}{3}x^3\right)\Big|_{-\lambda}^{0} + \frac{1}{\lambda^2}\left(-\frac{1}{4}x^4 + \frac{\lambda}{3}x^3\right)\Big|_0^{\lambda} \\ &= \frac{\lambda^2}{6} \end{aligned}$$

Hence, with $\sigma^2 = 1$, we obtain $\lambda = \sqrt{6}$ and $H(X) = \ln(6) - \ln(\sqrt{6}) + 1/2 = 1.7925$ bits /source output. A plot of the lower and upper bound of $R(D)$ is given in the next figure.



Problem 3.29 :

$$\sigma^2 = E[X^2(t)] = R_X(\tau)|_{\tau=0} = \frac{A^2}{2}$$

Hence,

$$\mathrm{SQNR} = 3 \cdot 4^{\nu} \overline{\breve{X}^2} = 3 \cdot 4^{\nu} \frac{\overline{X^2}}{x_{\max}^2} = 3 \cdot 4^{\nu} \frac{A^2}{2A^2}$$

With SQNR = 60 dB, we obtain

$$10 \log_{10} \left(\frac{3 \cdot 4^q}{2} \right) = 60 \Longrightarrow q = 9.6733$$

The smallest integer larger that q is 10. Hence, the required number of quantization levels is $\nu = 10$.

Problem 3.30 :

(a)

$$H(X|G) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x,g) \log p(x|g) dx dg$$

But X, G are independent, so : $p(x,g) = p(x)p(g),\ p(x|g) = p(x)$.Hence :

$$\begin{aligned} H(X|G) &= -\int_{-\infty}^{\infty} p(g) \left[\int_{-\infty}^{\infty} p(x) \log p(x) dx\right] dg \\ &= -\int_{-\infty}^{\infty} p(g) H(X) dg \\ &= H(X) = \tfrac{1}{2} \log(2\pi e \sigma_x^2) \end{aligned}$$

where the last equality stems from the Gaussian pdf of X.

(b)

$$I(X;Y) = H(Y) - H(Y|X)$$

Since Y is the sum of two independent, zero-mean Gaussian r.v's , it is also a zero-mean Gaussian r.v. with variance : $\sigma_y^2 = \sigma_x^2 + \sigma_n^2$. Hence : $H(Y) = \frac{1}{2} \log\left(2\pi e \left(\sigma_x^2 + \sigma_n^2\right)\right)$. Also, since $y = x + g$:

$$p(y|x) = p_g(y-x) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-\frac{(y-x)^2}{2\sigma_n^2}}$$

Hence :

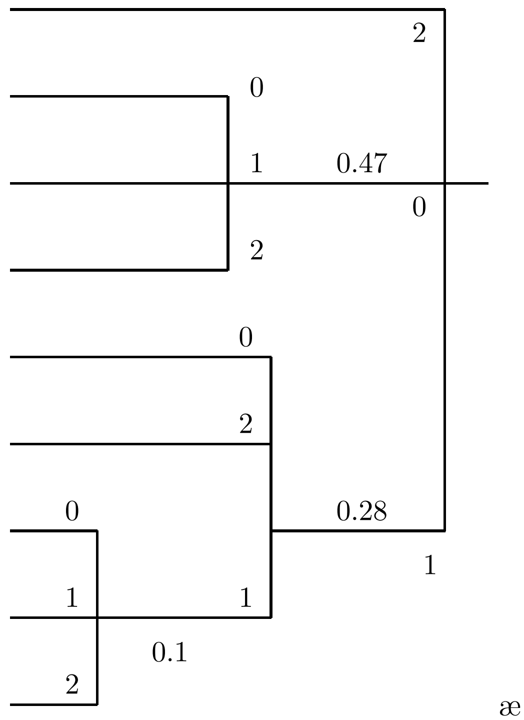
$$\begin{aligned} H(Y|X) &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(y|x) dx dy \\ &= -\int_{-\infty}^{\infty} p(x) \log e \int_{-\infty}^{\infty} p(y|x) \ln\left(\frac{1}{\sqrt{2\pi}\sigma_n} \exp(-\frac{(y-x)^2}{2\sigma_n^2})\right) dy dx \\ &= \int_{-\infty}^{\infty} p(x) \log e \left[\int_{-\infty}^{\infty} p_g(y-x) \left(\ln(\sqrt{2\pi}\sigma_n) + \frac{(y-x)^2}{2\sigma_n^2}\right) dy\right] dx \\ &= \int_{-\infty}^{\infty} p(x) \log e \left[\ln(\sqrt{2\pi}\sigma_n) + \frac{1}{2\sigma_n^2}\sigma_n^2\right] dx \\ &= \left[\log(\sqrt{2\pi}\sigma_n) + \frac{1}{2}\log e\right] \int_{-\infty}^{\infty} p(x) dx \\ &= \frac{1}{2} \log\left(2\pi e \sigma_n^2\right) \quad (= H(G)) \end{aligned}$$

where we have used the fact that : $\int_{-\infty}^{\infty} p_g(y-x) dy = 1,\quad \int_{-\infty}^{\infty} (y-x)^2 p_g(y-x) dy = E\left[G^2\right] = \sigma_n^2$. From $H(Y),\ H(Y|X)$:

$$I(X;Y) = H(Y) - H(Y|X) = \frac{1}{2}\log\left(2\pi e(\sigma_x^2 + \sigma_n^2)\right) - \frac{1}{2}\log\left(2\pi e \sigma_n^2\right) = \frac{1}{2}\log\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right)$$

Problem 3.31 :

Codeword	Letter	Probability
2	x_1	0.25
00	x_2	0.20
01	x_3	0.15
02	x_4	0.12
10	x_5	0.10
12	x_6	0.08
110	x_7	0.05
111	x_8	0.05
	x_9	0



$$\bar{R} = 1.85 \text{ ternary symbols/letter}$$

Problem 3.32 :

Given $(n_1, n_2, n_3, n_4) = (1, 2, 2, 3)$ we have :

$$\sum_{k=1}^{4} 2^{-n_k} = 2^{-1} + 2^{-2} + 2^{-2} + 2^{-3} = \frac{9}{8} > 1$$

Since the Craft inequality is not satisfied, a binary code with code word lengths $(1, 2, 2, 3)$ that satisfies the prefix condition does not exist.

Problem 3.33 :

$$\sum_{k=1}^{2^n} 2^{-n_k} = \sum_{k=1}^{2^n} 2^{-n} = 2^n 2^{-n} = 1$$

Therefore the Kraft inequality is satisfied.

Problem 3.34 :

$$p(\mathbf{X}) = \frac{1}{(2\pi)^{n/2}|M|^{1/2}} e^{-\frac{1}{2}\mathbf{X'M^{-1}X}}$$

$$H(\mathbf{X}) = -\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} p(\mathbf{X}) \log p(\mathbf{X}) d\mathbf{X}$$

But :

$$\log p(\mathbf{X}) = -\frac{1}{2}\log(2\pi)^n |M| - \left(\frac{1}{2}\log e\right) \mathbf{X'M^{-1}X}$$

and

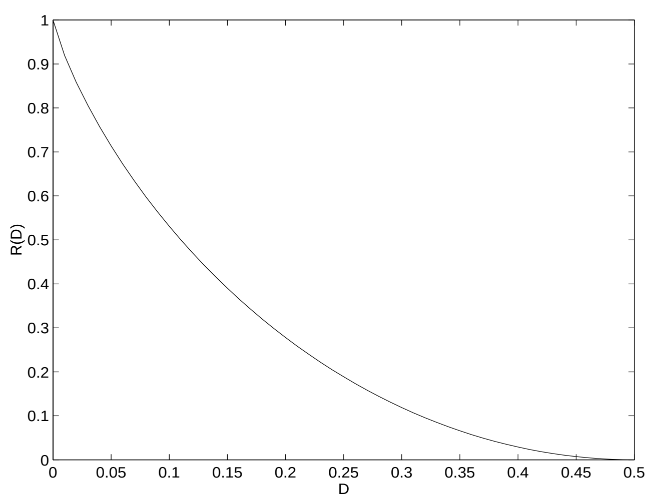
$$\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left(\frac{1}{2}\log e\right) \mathbf{X'M^{-1}X}\; p(\mathbf{X}) d\mathbf{X} = \frac{n}{2}\log e$$

Hence :

$$\begin{aligned} H(\mathbf{X}) &= \tfrac{1}{2}\log(2\pi)^n |M| + \tfrac{1}{2}\log e^n \\ &= \tfrac{1}{2}\log(2\pi e)^n |M| \end{aligned}$$

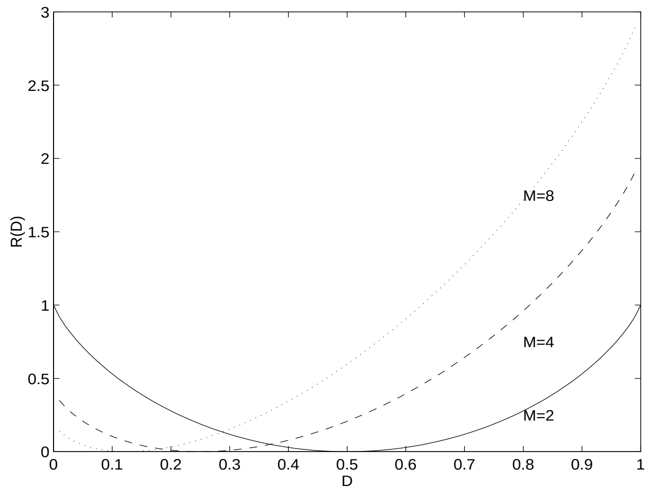
Problem 3.35 :

$$R(D) = 1 + D\log D + (1-D)\log(1-D), \quad 0 \le D = P_e \le 1/2$$



Problem 3.36 :

$$R(D) = \log M + D\log D + (1-D)\log\frac{(1-D)}{M-1}$$



Problem 3.37 :

$$d_W(\mathbf{X}, \tilde{\mathbf{X}}) = (\mathbf{X} - \tilde{\mathbf{X}})'\mathbf{W}(\mathbf{X} - \tilde{\mathbf{X}})$$

Let $\mathbf{W} = \mathbf{P}'\mathbf{P}$. Then :

$$\begin{aligned} d_W(\mathbf{X}, \tilde{\mathbf{X}}) &= (\mathbf{X} - \tilde{\mathbf{X}})'\mathbf{P}'\mathbf{P}(\mathbf{X} - \tilde{\mathbf{X}}) \\ &= \left(\mathbf{P}(\mathbf{X} - \tilde{\mathbf{X}})\right)'\mathbf{P}(\mathbf{X} - \tilde{\mathbf{X}}) \\ &= \frac{1}{n}\left(\mathbf{Y} - \tilde{\mathbf{Y}}\right)'\left(\mathbf{Y} - \tilde{\mathbf{Y}}\right) \end{aligned}$$

where by definition : $\mathbf{Y} = \sqrt{n}\mathbf{P}\mathbf{X}$, $\tilde{\mathbf{Y}} = \sqrt{n}\mathbf{P}\tilde{\mathbf{X}}$. Hence : $d_W(\mathbf{X}, \tilde{\mathbf{X}}) = d_2(\mathbf{Y}, \tilde{\mathbf{Y}})$.

Problem 3.38 :

(a) The first order predictor is : $\hat{x}(n) = a_{11}x(n-1)$. The coefficient a_{11} that minimizes the MSE is found from the orthogonality of the prediction error to the prediction data :

$$\begin{aligned} &E\left[e(n)x(n-1)\right] = 0 \Rightarrow \\ &E\left[\left(x(n) - a_{11}x(n-1)\right)x(n-1)\right] = 0 \Rightarrow \\ &\phi(1) - a_{11}\phi(0) = 0 \Rightarrow a_{11} = \phi(1)/\phi(0) = 1/2 \end{aligned}$$

The minimum MSE is : $\epsilon_1 = \phi(0)\,(1-a_{11}^2) = 3/4$

(b) For the second order predictor : $\hat{x}(n) = a_{21}x(n-1) + a_{22}x(n-2)$. Following the Levinson-Durbin algorithm (Eqs 3-5-25) :

$$a_{22} = \frac{\phi(2) - \sum_{k=1}^{1} a_{1k}\phi(2-k)}{\epsilon_1} = \frac{0 - \frac{1}{2}\frac{1}{2}}{3/4} = -1/3$$

$$a_{21} = a_{11} - a_{22}a_{11} = 2/3$$

The minimum MSE is :

$$\epsilon_2 = \epsilon_1\,(1-a_{22})^2 = 2/3$$

Problem 3.39 :

$$p(x_1,x_2) = \left\{ \begin{array}{cc} \frac{15}{7ab}, & x_1, x_2\ \in C \\ 0, & \text{o.w} \end{array} \right\}$$

If x_1, x_2 are quantized separately by using uniform intervals of length Δ, the number of levels needed is $L_1 = \frac{a}{\Delta},\ L_2 = \frac{b}{\Delta}$. The number of bits is :

$$R_x = R_1 + R_2 = \log L_1 + \log L_2 = \log\frac{ab}{\Delta^2}$$

By using vector quantization with squares having area Δ^2, we have $L'_x = \frac{7ab}{15\Delta^2}$ and $R'_x = \log L'_x = \log\frac{7ab}{15\Delta^2}$ bits. The difference in bit rate is :

$$R_x - R'_x = \log\frac{ab}{\Delta^2} - \log\frac{7ab}{15\Delta^2} = \log\frac{15}{7} = 1.1 \text{ bits/output sample}$$

for all $a, b > 0$.

Problem 3.40 :

(a) The area between the two squares is $4\times 4 - 2\times 2 = 12$. Hence, $p_{X,Y}(x,y) = \frac{1}{12}$. The marginal probability $p_X(x)$ is given by $p_X(x) = \int_{-2}^{2} p_{X,Y}(x,y)dy$. If $-2 \leq X < -1$, then

$$p_X(x) = \int_{-2}^{2} p_{X,Y}(x,y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

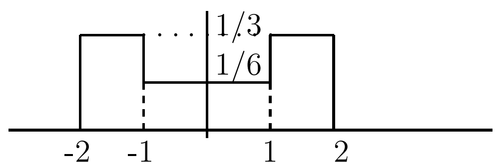
If $-1 \leq X < 1$, then

$$p_X(x) = \int_{-2}^{-1} \frac{1}{12} dy + \int_{1}^{2} \frac{1}{12} dy = \frac{1}{6}$$

Finally, if $1 \leq X \leq 2$, then

$$p_X(x) = \int_{-2}^{2} p_{X,Y}(x,y) dy = \frac{1}{12} y \Big|_{-2}^{2} = \frac{1}{3}$$

The next figure depicts the marginal distribution $p_X(x)$.



Similarly we find that

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y < -1 \\ \frac{1}{6} & -1 \leq y < -1 \\ \frac{1}{3} & 1 \leq y \leq 2 \end{cases}$$

(b) The quantization levels $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are set to $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$ respectively. The resulting distortion is

$$\begin{aligned} D_X &= 2 \int_{-2}^{-1} (x + \frac{3}{2})^2 p_X(x) dx + 2 \int_{-1}^{0} (x + \frac{1}{2})^2 p_X(x) dx \\ &= \frac{2}{3} \int_{-2}^{-1} (x^2 + 3x + \frac{9}{4}) dx + \frac{2}{6} \int_{-1}^{0} (x^2 + x + \frac{1}{4}) dx \\ &= \frac{2}{3} \left(\frac{1}{3} x^3 + \frac{3}{2} x^2 + \frac{9}{4} x \right) \Big|_{-2}^{-1} + \frac{2}{6} \left(\frac{1}{3} x^3 + \frac{1}{2} x^2 + \frac{1}{4} x \right) \Big|_{-1}^{0} \\ &= \frac{1}{12} \end{aligned}$$

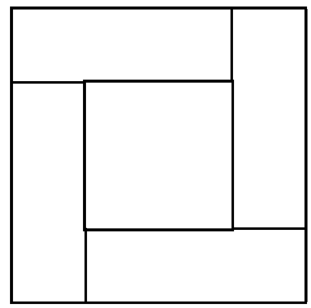
The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

whereas the resulting number of bits per (X, Y) pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

(c) Suppose that we divide the region over which $p(x, y) \neq 0$ into L equal subregions. The case of $L = 4$ is depicted in the next figure.



For each subregion the quantization output vector $(\hat{x}, \hat{y})$ is the centroid of the corresponding rectangle. Since, each subregion has the same shape (uniform quantization), a rectangle with width equal to one and length $12/L$, the distortion of the vector quantizer is

$$\begin{aligned} D &= \int_0^1 \int_0^{\frac{12}{L}} [(x,y) - (\frac{1}{2}, \frac{12}{2L})]^2 \frac{L}{12} dxdy \\ &= \frac{L}{12} \int_0^1 \int_0^{\frac{12}{L}} \left[(x - \frac{1}{2})^2 + (y - \frac{12}{2L})^2 \right] dxdy \\ &= \frac{L}{12} \left[\frac{12}{L}\frac{1}{12} + \frac{12^3}{L^3}\frac{1}{12} \right] = \frac{1}{12} + \frac{12}{L^2} \end{aligned}$$

If we set $D = \frac{1}{6}$, we obtain

$$\frac{12}{L^2} = \frac{1}{12} \Longrightarrow L = \sqrt{144} = 12$$

Thus, we have to divide the area over which $p(x, y) \neq 0$, into 12 equal subregions in order to achieve the same distortion. In this case the resulting number of bits per source output pair (X, Y) is $R = \log_2 12 = 3.585$.

Problem 3.41 :

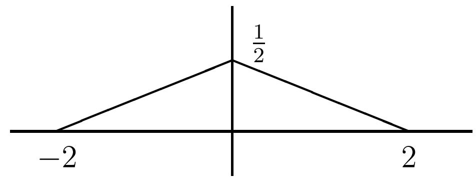
(a) The joint probability density function is $p_{XY}(x, y) = \frac{1}{(2\sqrt{2})^2} = \frac{1}{8}$. The marginal distribution $p_X(x)$ is $p_X(x) = \int_y p_{XY}(x, y)dy$. If $-2 \leq x \leq 0$,then

$$p_X(x) = \int_{-x-2}^{x+2} p_{X,Y}(x, y)dy = \frac{1}{8} y|_{-x-2}^{x+2} = \frac{x+2}{4}$$

If $0 \leq x \leq 2$,then

$$p_X(x) = \int_{x-2}^{-x+2} p_{X,Y}(x, y)dy = \frac{1}{8} y|_{x-2}^{-x+2} = \frac{-x+2}{4}$$

The next figure depicts $p_X(x)$.



From the symmetry of the problem we have

$$p_Y(y) = \begin{cases} \frac{y+2}{4} & -2 \le y < 0 \\ \frac{-y+2}{4} & 0 \le y \le 2 \end{cases}$$

(b)

$$\begin{aligned}
D_X &= 2\int_{-2}^{-1}(x+\frac{3}{2})^2 p_X(x)dx + 2\int_{-1}^{0}(x+\frac{1}{2})^2 p_X(x)dx \\
&= \frac{1}{2}\int_{-2}^{-1}(x+\frac{3}{2})^2(x+2)dx + \frac{1}{2}\int_{-1}^{0}(x+\frac{1}{2})^2(-x+2)dx \\
&= \frac{1}{2}\left(\frac{1}{4}x^4+\frac{5}{3}x^3+\frac{33}{8}x^2+\frac{9}{2}x\right)\Big|_{-2}^{-1} + \frac{1}{2}\left(\frac{1}{4}x^4+x^3+\frac{9}{8}x^2+\frac{1}{2}x\right)\Big|_{-1}^{0} \\
&= \frac{1}{12}
\end{aligned}$$

The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12}+\frac{1}{12} = \frac{1}{6}$$

whereas the required number of bits per source output pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

(c) We divide the square over which $p(x,y) \neq 0$ into $2^4 = 16$ equal square regions. The area of each square is $\frac{1}{2}$ and the resulting distortion

$$\begin{aligned}
D &= \frac{16}{8}\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}\left[(x-\frac{1}{2\sqrt{2}})^2+(y-\frac{1}{2\sqrt{2}})^2\right]dxdy \\
&= 4\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}(x-\frac{1}{2\sqrt{2}})^2 dxdy \\
&= \frac{4}{\sqrt{2}}\int_0^{\frac{1}{\sqrt{2}}}(x^2+\frac{1}{8}-\frac{x}{\sqrt{2}})dx \\
&= \frac{4}{\sqrt{2}}\left(\frac{1}{3}x^3+\frac{1}{8}x-\frac{1}{2\sqrt{2}}x^2\right)\Big|_0^{\frac{1}{\sqrt{2}}} \\
&= \frac{1}{12}
\end{aligned}$$

Hence, using vector quantization and the same rate we obtain half the distortion.

CHAPTER 4

Problem 4.1 :

(a)

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(a)}{t-a}da$$

Hence :

$$\begin{array}{rcl} -\hat{x}(-t) & = & -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(a)}{-t-a}da \\ & = & -\frac{1}{\pi}\int_{\infty}^{-\infty}\frac{x(-b)}{-t+b}(-db) \\ & = & -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(b)}{-t+b}db \\ & = & \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(b)}{t-b}db = \hat{x}(t) \end{array}$$

where we have made the change of variables : $b = -a$ and used the relationship : $x(b) = x(-b)$.

(b) In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

(c) $x(t) = \cos\omega_0 t$, so its Fourier transform is : $X(f) = \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right],\ f_0 = 2\pi\omega_0$. Exploiting the phase-shifting property (4-1-7) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2}\left[-j\delta(f-f_0) + j\delta(f+f_0)\right] = \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] = F^{-1}\left\{\sin 2\pi f_0 t\right\}$$

Hence, $\hat{x}(t) = \sin\omega_0 t$.

(d) In a similar way to part (c) :

$$x(t) = \sin\omega_0 t \Rightarrow X(f) = \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] \Rightarrow \hat{X}(f) = \frac{1}{2}\left[-\delta(f-f_0) - \delta(f+f_0)\right]$$

$$\Rightarrow \hat{X}(f) = -\frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] = -F^{-1}\left\{\cos 2\pi\omega_0 t\right\} \Rightarrow \hat{x}(t) = -\cos\omega_0 t$$

(e) The positive frequency content of the new signal will be : $(-j)(-j)X(f) = -X(f),\ f > 0$, while the negative frequency content will be : $j \cdot jX(f) = -X(f),\ f < 0$. Hence, since $\hat{\hat{X}}(f) = -X(f)$, we have : $\hat{\hat{x}}(t) = -x(t)$.

(f) Since the magnitude response of the Hilbert transformer is characterized by : $|H(f)| = 1$, we have that : $\left|\hat{X}(f)\right| = |H(f)|\,|X(f)| = |X(f)|$. Hence :

$$\int_{-\infty}^{\infty} \left|\hat{X}(f)\right|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

(g) From parts (a) and (b) above, we note that if $x(t)$ is even, $\hat{x}(t)$ is odd and vice-versa. Therefore, $x(t)\hat{x}(t)$ is always odd and hence : $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.

Problem 4.2 :

We have :

$$\hat{x}(t) = h(t) * x(t)$$

where $h(t) = \frac{1}{\pi t}$ and $H(f) = \left\{ \begin{array}{cc} -j, & \text{f} > 0 \\ j, & \text{f} < 0 \end{array} \right\}$. Hence :

$$\Phi_{\hat{x}\hat{x}}(f) = \Phi_{xx}(f) \, |H(f)|^2 = \Phi_{xx}(f)$$

and its inverse Fourier transform :

$$\phi_{\hat{x}\hat{x}}(\tau) = \phi_{xx}(\tau)$$

Also :

$$\begin{aligned}
\phi_{x\hat{x}}(\tau) &= E\left[x(t+\tau)\hat{x}(t)\right] \\
&= \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{E[x(t+\tau)x(a)]}{t-a} da \\
&= \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{\phi_{xx}(t+\tau-a)}{t-a} da \\
&= -\tfrac{1}{\pi}\int_{\infty}^{-\infty} \tfrac{\phi_{xx}(b)}{b-\tau} db \\
&= \tfrac{1}{\pi}\int_{-\infty}^{\infty} \tfrac{\phi_{xx}(b)}{\tau-b} db = -\phi_{xx}(\tau)
\end{aligned}$$

Problem 4.3 :

(a)

$$\begin{aligned} E\left[z(t)z(t+\tau)\right] &= E\left[\left\{x(t+\tau)+jy(t+t)\right\}\left\{x(t)+jy(t)\right\}\right] \\ &= E\left[x(t)x(t+\tau)\right]-E\left[y(t)y(t+\tau)\right]+jE\left[x(t)y(t+\tau)\right] \\ &\quad +E\left[y(t)x(t+\tau)\right] \\ &= \phi_{xx}(\tau)-\phi_{yy}(\tau)+j\left[\phi_{yx}(\tau)+\phi_{xy}(\tau)\right] \end{aligned}$$

But $\phi_{xx}(\tau)=\phi_{yy}(\tau)$and $\phi_{yx}(\tau)=-\phi_{xy}(\tau)$. Therefore :

$$E\left[z(t)z(t+\tau)\right]=0$$

(b)

$$V=\int_0^T z(t)dt$$

$$E\left(V^2\right)=\int_0^T\int_0^T E\left[z(a)z(b)\right]dadb=0$$

from the result in (a) above. Also :

$$\begin{aligned} E\left(VV^*\right) &= \int_0^T\int_0^T E\left[z(a)z^*(b)\right]dadb \\ &= \int_0^T\int_0^T 2N_0\delta(a-b)dadb \\ &= \int_0^T 2N_0da=2N_0T \end{aligned}$$

Problem 4.4 :

$$\begin{aligned} E\left[x(t+\tau)x(t)\right] &= A^2E\left[\sin\left(2\pi f_c(t+\tau)+\theta\right)\sin\left(2\pi f_ct+\theta\right)\right] \\ &= \frac{A^2}{2}\cos 2\pi f_c\tau-\frac{A^2}{2}E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] \end{aligned}$$

where the last equality follows from the trigonometric identity :
$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$. But :

$$\begin{aligned} E\left[\cos\left(2\pi f_c(2t+\tau)+2\theta\right)\right] &= \int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)p(\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos\left(2\pi f_c(2t+\tau)+2\theta\right)d\theta=0 \end{aligned}$$

Hence :

$$E\left[x(t+\tau)x(t)\right]=\frac{A^2}{2}\cos 2\pi f_c\tau$$

Problem 4.5 :

We know from Fourier transform properties that if a signal $x(t)$ is real-valued then its Fourier transform satisfies : $X(-f) = X^*(f)$ (Hermitian property). Hence the condition under which $s_l(t)$ is real-valued is : $S_l(-f) = S_l^*(f)$ or going back to the bandpass signal $s(t)$ (using 4-1-8):

$$S_+(f_c - f) = S_+^*(f_c + f)$$

The last condition shows that in order to have a real-valued lowpass signal $s_l(t)$, the positive frequency content of the corresponding bandpass signal must exhibit hermitian symmetry around the center frequency f_c. In general, bandpass signals do not satisfy this property (they have Hermitian symmetry around $f = 0$), hence, the lowpass equivalent is generally complex-valued.

Problem 4.6 :

A well-known result in estimation theory based on the minimum mean-squared-error criterion states that the minimum of $\mathcal{E}_e$ is obtained when the error is orthogonal to each of the functions in the series expansion. Hence :

$$\int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right] f_n^*(t)dt = 0, \qquad n = 1, 2, ..., K \tag{1}$$

since the functions $\{f_n(t)\}$ are orthonormal, only the term with $k = n$ will remain in the sum, so :

$$\int_{-\infty}^{\infty} s(t)f_n^*(t)dt - s_n = 0, \qquad n = 1, 2, ..., K$$

or:

$$s_n = \int_{-\infty}^{\infty} s(t)f_n^*(t)dt \qquad n = 1, 2, ..., K$$

The corresponding residual error $\mathcal{E}_e$ is :

$$\begin{aligned}
\mathcal{E}_{\min} &= \int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right]\left[s(t) - \sum_{n=1}^{K} s_n f_n(t)\right]^* dt \\
&= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty}\sum_{k=1}^{K} s_k f_k(t)s^*(t)dt - \sum_{n=1}^{K} s_n^* \int_{-\infty}^{\infty}\left[s(t) - \sum_{k=1}^{K} s_k f_k(t)\right] f_n^*(t)dt \\
&= \int_{-\infty}^{\infty} |s(t)|^2 dt - \int_{-\infty}^{\infty}\sum_{k=1}^{K} s_k f_k(t)s^*(t)dt \\
&= \mathcal{E}_s - \sum_{k=1}^{K} |s_k|^2
\end{aligned}$$

where we have exploited relationship (1) to go from the second to the third step in the above calculation.

Note : Relationship (1) can also be obtained by simple differentiation of the residual error with respect to the coefficients $\{s_n\}$. Since s_n is, in general, complex-valued $s_n = a_n + jb_n$ we have to differentiate with respect to both real and imaginary parts :

$$\frac{d}{da_n}\mathcal{E}_e = \frac{d}{da_n}\int_{-\infty}^{\infty}\left[s(t)-\sum_{k=1}^{K}s_kf_k(t)\right]\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]^* dt = 0$$

$$\Rightarrow -\int_{-\infty}^{\infty}a_nf_n(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]^* + a_n^*f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]dt = 0$$

$$\Rightarrow -2a_n\int_{-\infty}^{\infty}Re\left\{f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]\right\}dt = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}Re\left\{f_n^*(t)\left[s(t)-\sum_{n=1}^{K}s_nf_n(t)\right]\right\}dt = 0, \qquad n = 1,2,...,K$$

where we have exploited the identity : $(x+x^*) = 2Re\{x\}$. Differentiation of $\mathcal{E}_e$ with respect to b_n will give the corresponding relationship for the imaginary part; combining the two we get (1).

Problem 4.7 :

The procedure is very similar to the one for the real-valued signals described in the book (pages 167-168). The only difference is that the projections should conform to the complex-valued vector space :

$$c_{12=}\int_{-\infty}^{\infty}s_2(t)f_1^*(t)dt$$

and, in general for the k-th function :

$$c_{ik} = \int_{-\infty}^{\infty}s_k(t)f_i^*(t)dt, \quad i = 1,2,...,k-1$$

Problem 4.8 :

For real-valued signals the correlation coefficients are given by : $\rho_{km} = \frac{1}{\sqrt{\mathcal{E}_k\mathcal{E}_m}}\int_{-\infty}^{\infty}s_k(t)s_m(t)dt$ and the Euclidean distances by : $d_{km}^{(e)} = \left\{\mathcal{E}_k+\mathcal{E}_m-2\sqrt{\mathcal{E}_k\mathcal{E}_m}\rho_{km}\right\}^{1/2}$. For the signals in this problem :

$$\mathcal{E}_1 = 2,\ \mathcal{E}_2 = 2,\ \mathcal{E}_3 = 3,\ \mathcal{E}_4 = 3$$

$$\begin{array}{lll} \rho_{12} = 0 & \rho_{13} = \frac{2}{\sqrt{6}} & \rho_{14} = -\frac{2}{\sqrt{6}} \\ \rho_{23} = 0 & \rho_{24} = 0 & \\ \rho_{34} = -\frac{1}{3} & & \end{array}$$

and:

$$
\begin{array}{lll}
d_{12}^{(e)} = 2 & d_{13}^{(e)} = \sqrt{2+3-2\sqrt{6}\frac{2}{\sqrt{6}}} = 1 & d_{14}^{(e)} = \sqrt{2+3+2\sqrt{6}\frac{2}{\sqrt{6}}} = 3 \\
d_{23}^{(e)} = \sqrt{2+3 =}\sqrt{5} & d_{24}^{(e)} = \sqrt{5} & \\
d_{34}^{(e)} = \sqrt{3+3+2*3\frac{1}{3}} = 2\sqrt{2} & &
\end{array}
$$

Problem 4.9 :

The energy of the signal waveform $s'_m(t)$ is :

$$
\begin{aligned}
\mathcal{E}' &= \int_{-\infty}^{\infty} |s'_m(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} s_m^2(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt \\
&\quad -\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt - \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt \\
&= \mathcal{E} + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\mathcal{E}\delta_{kl} - \frac{2}{M}\mathcal{E} \\
&= \mathcal{E} + \frac{1}{M}\mathcal{E} - \frac{2}{M}\mathcal{E} = \left(\frac{M-1}{M}\right)\mathcal{E}
\end{aligned}
$$

The correlation coefficient is given by :

$$
\begin{aligned}
\rho_{mn} &= \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty} s'_m(t)s'_n(t)dt = \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty}\left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M}s_k(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M}s_l(t)\right)dt \\
&= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty} s_m(t)s_n(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt\right) \\
&\quad -\frac{1}{\mathcal{E}'}\left(\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_n(t)s_k(t)dt + \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt\right) \\
&= \frac{\frac{1}{M^2}M\mathcal{E} - \frac{1}{M}\mathcal{E} - \frac{1}{M}\mathcal{E}}{\frac{M-1}{M}\mathcal{E}} = -\frac{1}{M-1}
\end{aligned}
$$

Problem 4.10 :

(a) To show that the waveforms $f_n(t)$, $n = 1, \ldots, 3$ are orthogonal we have to prove that:

$$
\int_{-\infty}^{\infty} f_m(t)f_n(t)dt = 0, \qquad m \neq n
$$

Clearly:

$$\begin{aligned}
c_{12} &= \int_{-\infty}^{\infty} f_1(t)f_2(t)dt = \int_0^4 f_1(t)f_2(t)dt \\
&= \int_0^2 f_1(t)f_2(t)dt + \int_2^4 f_1(t)f_2(t)dt \\
&= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) \\
&= 0
\end{aligned}$$

Similarly:

$$\begin{aligned}
c_{13} &= \int_{-\infty}^{\infty} f_1(t)f_3(t)dt = \int_0^4 f_1(t)f_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

and :

$$\begin{aligned}
c_{23} &= \int_{-\infty}^{\infty} f_2(t)f_3(t)dt = \int_0^4 f_2(t)f_3(t)dt \\
&= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\
&= 0
\end{aligned}$$

Thus, the signals $f_n(t)$ are orthogonal. It is also straightforward to prove that the signals have unit energy :

$$\int_{-\infty}^{\infty} |f_i(t)|^2 dt = 1, \quad i = 1, 2, 3$$

Hence, they are orthonormal.

(b) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)f_n(t)dt, \qquad n = 1, 2, 3$$

$$\begin{aligned}
x_1 &= \int_0^4 x(t)f_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\
x_2 &= \int_0^4 x(t)f_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\
x_3 &= \int_0^4 x(t)f_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0
\end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal wavaforms $f_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 4.11 :

(a) As an orthonormal set of basis functions we consider the set

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad f_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

(b) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

(c) The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix}\right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}\right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix}\right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix}\right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix}\right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 4.12 :

As a set of orthonormal functions we consider the waveforms

$$f_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad f_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad f_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 4.13 :

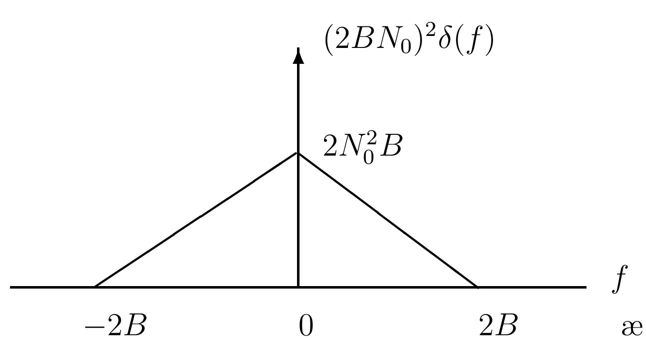
The power spectral density of $X(t)$ corresponds to : $\phi_{xx}(t) = 2BN_0 \frac{\sin 2\pi Bt}{2\pi Bt}$. From the result of Problem 2.14 :

$$\phi_{yy}(\tau) = \phi_{xx}^2(0) + 2\phi_{xx}^2(\tau) = (2BN_0)^2 + 8B^2N_0^2 \left(\frac{\sin 2\pi Bt}{2\pi Bt}\right)^2$$

Also :

$$\Phi_{yy}(f) = \phi_{xx}^2(0)\delta(f) + 2\Phi_{xx}(f) * \Phi_{xx}(f)$$

The following figure shows the power spectral density of $Y(t)$:



Problem 4.14 :

$$u(t) = \sum_n \left[a_n g(t-2nT) - jb_n g(t-2nT-T)\right]$$

(a) Since the signaling rate is 1/2T for each sequence and since $g(t)$ has duration 2T, for any time instant only $g(t-2nT)$ and $g(t-2nT-T)$ or $g(t-2nT+T)$ will contribute to $u(t)$. Hence, for $2nT \le t \le 2nT+T$:

$$\begin{aligned} |u(t)|^2 &= |a_n g(t-2nT) - jb_n g(t-2nT+T)|^2 \\ &= a_n^2 g^2(t-2nT) + b_n^2 g^2(t-2nT+T) \\ &= g^2(t-2nT) + g^2(t-2nT+T) = \sin^2\frac{\pi t}{2T} + \sin^2\frac{\pi(t+T)}{2T} \\ &= \sin^2\frac{\pi t}{2T} + \cos^2\frac{\pi t}{2T} = 1, \quad \forall\, t \end{aligned}$$

(b) The power density spectrum is :

$$\Phi_{uu}(f) = \frac{1}{T}\,|G(f)|^2$$

where $G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt = \int_0^{2T} \sin\frac{\pi t}{2T}\exp(-j2\pi ft)dt$. By using the trigonometric identity $\sin x = \frac{\exp(jx)-\exp(-jx)}{2j}$ it is easily shown that :

$$G(f) = \frac{4T}{\pi}\frac{\cos 2\pi Tf}{1-16T^2f^2}e^{-j2\pi fT}$$

Hence :

$$\begin{aligned} G(f) &= \left(\frac{4T}{\pi}\right)^2 \frac{\cos^2 2\pi Tf}{(1-16T^2f^2)^2} \\ \Phi_{uu}(f) &= \frac{1}{T}\left(\frac{4T}{\pi}\right)^2 \frac{\cos^2 2\pi Tf}{(1-16T^2f^2)^2} \\ &= \frac{16T}{\pi^2}\frac{\cos^2 2\pi Tf}{(1-16T^2f^2)^2} \end{aligned}$$

(c) The above power density spectrum is identical to that for the MSK signal. Therefore, the MSK signal can be generated as a staggered four phase PSK signal with a half-period sinusoidal pulse for $g(t)$.

Problem 4.15:

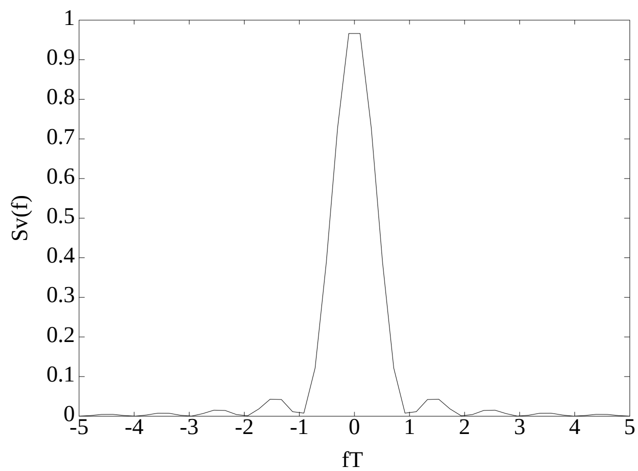
We have that $\Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2 \Phi_{ii}(f)$ But $\mathrm{E}(I_n) = 0,\ E\left(\left|I_n\right|^2\right) = 1$, hence : $\phi_{ii}(m) = \left\{\begin{array}{ll} 1, & m=0 \\ 0, & m \neq 0 \end{array}\right\}$. Therefore : $\Phi_{ii}(f) = 1 \Rightarrow \Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2$.

(a) For the rectangular pulse :

$$G(f) = AT\frac{\sin \pi fT}{\pi fT}e^{-j2\pi fT/2} \Rightarrow \left|G(f)\right|^2 = A^2T^2\frac{\sin^2 \pi fT}{(\pi fT)^2}$$

where the factor $e^{-j2\pi fT/2}$ is due to the $T/2$ shift of the rectangular pulse from the center $t=0$. Hence :

$$\Phi_{uu}(f) = A^2T\frac{\sin^2 \pi fT}{(\pi fT)^2}$$

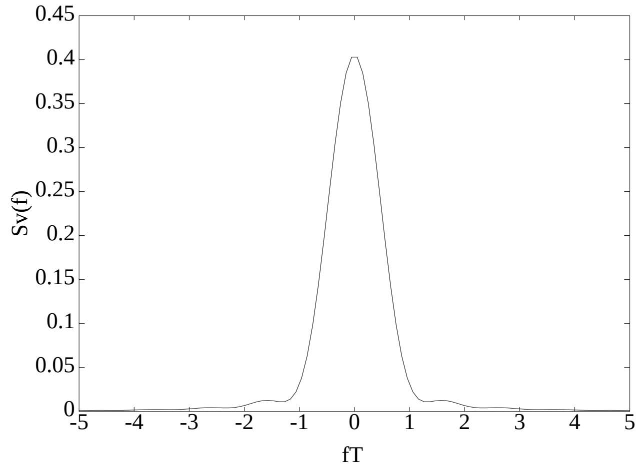


(b) For the sinusoidal pulse : $G(f) = \int_0^T \sin\frac{\pi t}{T}\exp(-j2\pi ft)dt$. By using the trigonometric identity $\sin x = \frac{\exp(jx)-\exp(-jx)}{2j}$ it is easily shown that :

$$G(f) = \frac{2AT}{\pi}\frac{\cos \pi Tf}{1-4T^2f^2}e^{-j2\pi fT/2} \Rightarrow \left|G(f)\right|^2 = \left(\frac{2AT}{\pi}\right)^2\frac{\cos^2 \pi Tf}{(1-4T^2f^2)^2}$$

Hence :

$$\Phi_{uu}(f) = \left(\frac{2A}{\pi}\right)^2 T\frac{\cos^2 \pi Tf}{(1-4T^2f^2)^2}$$



(c) The 3-db frequency for (a) is :

$$\frac{\sin^2 \pi f_{3db} T}{(\pi f_{3db} T)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.44}{T}$$

(where this solution is obtained graphically), while the 3-db frequency for the sinusoidal pulse on (b) is :

$$\frac{\cos^2 \pi T f}{(1 - 4T^2 f^2)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.59}{T}$$

The rectangular pulse spectrum has the first spectral null at $f = 1/T$, whereas the spectrum of the sinusoidal pulse has the first null at $f = 3/2T = 1.5/T$. Clearly the spectrum for the rectangular pulse has a narrower main lobe. However, it has higher sidelobes.

Problem 4.16 :

$$u(t) = X \cos 2\pi f t - Y \sin 2\pi f t$$
$$E[u(t)] = E(X) \cos 2\pi f t - E(Y) \sin 2\pi f t$$

and :

$$\begin{aligned}
\phi_{uu}(t, t+\tau) &= E\left\{[X \cos 2\pi f t - Y \sin 2\pi f t][X \cos 2\pi f(t+\tau) - Y \sin 2\pi f(t+\tau)]\right\} \\
&= E\left(X^2\right)[\cos 2\pi f(2t+\tau) + \cos 2\pi f\tau] + E\left(Y^2\right)[-\cos 2\pi f(2t+\tau) + \cos 2\pi f\tau] \\
&\quad - E(XY) \sin 2\pi f(2t+\tau)
\end{aligned}$$

For $u(t)$ to be wide-sense stationary, we must have : $E[u(t)]$ =constant and $\phi_{uu}(t,t+\tau) = \phi_{uu}(\tau)$. We note that if $E(X) = E(Y) = 0$, and $E(XY) = 0$ and $E(X^2) = E(Y^2)$, then the above requirements for WSS hold; hence these conditions are necessary. Conversely, if any of the above conditions does not hold, then either $E[u(t)] \neq$constant, or $\phi_{uu}(t,t+\tau) \neq \phi_{uu}(\tau)$. Hence, the conditions are also necessary.

Problem 4.17 :

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \left\{ \begin{array}{cc} -1/\sqrt{3}, & 0 \le t \le 3 \\ 0, & \text{o.w.} \end{array} \right\}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t) g_4(t) dt = -1/\sqrt{3} \Rightarrow g_3'(t) = s_3(t) - c_{43} g_4(t) = \left\{ \begin{array}{cc} 2/3, & 0 \le t \le 2 \\ -4/3, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{array} \right\}$$

Hence :

$$g_3(t) = \frac{g_3'(t)}{\sqrt{E_3}} = \left\{ \begin{array}{cc} 1/\sqrt{6}, & 0 \le t \le 2 \\ -2/\sqrt{6}, & 2 \le t \le 3 \\ 0, & \text{o.w} \end{array} \right\}$$

where E_3 denotes the energy of $g_3'(t)$: $E_3 = \int_0^3 (g_3'(t))^2 dt = 8/3$.
For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t) g_4(t) dt = 0 \quad \text{and} \quad \mathrm{c}_{32} = \int_{-\infty}^{\infty} s_2(t) g_3(t) dt = 0$$

Hence :

$$g_2'(t) = s_2(t) - c_{42} g_4(t) - c_{32} g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g_2'(t)}{\sqrt{\mathcal{E}_2}} = \left\{ \begin{array}{cc} 1/\sqrt{2}, & 0 \le t \le 1 \\ -1/\sqrt{2}, & 1 \le t \le 2 \\ 0, & \text{o.w} \end{array} \right\}$$

where : $\mathcal{E}_2 = \int_0^2 (s_2(t))^2 dt = 2$.
Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t) g_4(t) dt = -2/\sqrt{3}, \ \mathrm{c}_{31} = \int_{-\infty}^{\infty} s_1(t) g_3(t) dt = 2/\sqrt{6}, \ c_{21} = 0$$

Hence :

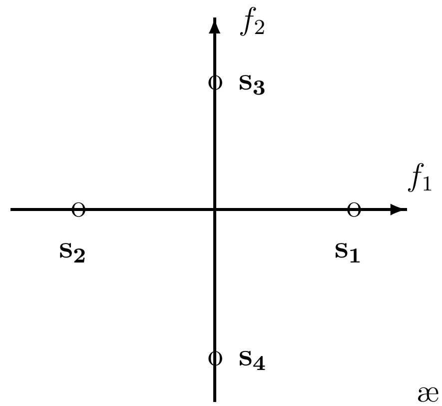
$$g_1'(t) = s_1(t) - c_{41} g_4(t) - c_{31} g_3(t) - c_{21} g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions $(g_2(t), g_3(t), g_4(t))$ the basis representation of the signals is :

$$\begin{array}{l} \mathbf{s}_4 = \left(0, 0, \sqrt{3}\right) \Rightarrow \mathcal{E}_4 = 3 \\ \mathbf{s}_3 = \left(0, \sqrt{8/3}, -1/\sqrt{3}\right) \Rightarrow \mathcal{E}_3 = 3 \\ \mathbf{s}_2 = \left(\sqrt{2}, 0, 0\right) \Rightarrow \mathcal{E}_2 = 2 \\ \mathbf{s}_1 = \left(2/\sqrt{6}, -2/\sqrt{3}, 0\right) \Rightarrow \mathcal{E}_1 = 2 \end{array}$$

Problem 4.18 :

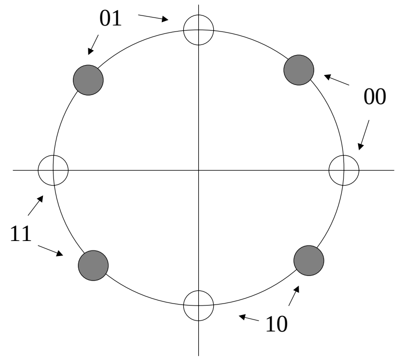
$$\begin{array}{l} \mathbf{s}_1 = \left(\sqrt{\mathcal{E}}, 0\right) \\ \mathbf{s}_2 = \left(-\sqrt{\mathcal{E}}, 0\right) \\ \mathbf{s}_3 = \left(0, \sqrt{\mathcal{E}}\right) \\ \mathbf{s}_4 = \left(0, -\sqrt{\mathcal{E}}\right) \end{array}$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

Problem 4.19 :

(a)(b) The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT \quad n = 0, 1, ...$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1, \quad n = 0, 1, ...$ are given with filled circles.

Problem 4.20 :

The autocorrelation function for $u_\Delta(t)$ is :

$$\begin{aligned}
\phi_{u_\Delta u_\Delta}(t) &= \tfrac{1}{2} E\left[u_\Delta(t+\tau) u_\Delta^*(t)\right] \\
&= \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left(I_m I_n^*\right) E\left[u(t+\tau-mT-\Delta) u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \phi_{ii}(m-n) E\left[u(t+\tau-mT-\Delta) u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} E\left[u(t+\tau-mT-nT-\Delta) u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \int_0^T \tfrac{1}{T} u(t+\tau-mT-nT-\Delta) u^*(t-nT-\Delta) d\Delta
\end{aligned}$$

Let $a = \Delta + nT,\ da = d\Delta,$ and $a \in (-\infty, \infty)$. Then :

$$\begin{aligned}
\phi_{u_\Delta u_\Delta}(t) &= \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} \tfrac{1}{T} u(t+\tau-mT-a) u^*(t-a) da \\
&= \tfrac{1}{2} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \tfrac{1}{T} \int_{-\infty}^{\infty} u(t+\tau-mT-a) u^*(t-a) da \\
&= \tfrac{1}{T} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \phi_{uu}(\tau - mT)
\end{aligned}$$

Thus we have obtained the same autocorrelation function as given by (4.4.11). Consequently the power spectral density of $u_\Delta(t)$ is the same as the one given by (4.4.12) :

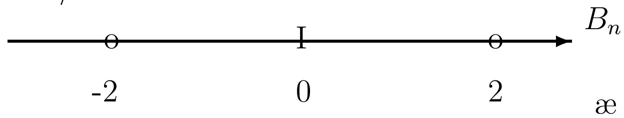
$$\Phi_{u_\Delta u_\Delta}(f) = \frac{1}{T} \left|G(f)\right|^2 \Phi_{ii}(f)$$

Problem 4.21 :

(a) $B_n = I_n + I_{n-1}$. Hence :

I_n	I_{n-1}	B_n
1	1	2
1	−1	0
−1	1	0
−1	−1	−2

The signal space representation is given in the following figure, with $P(B_n = 2) = P(B_n = -2) = 1/4, \quad P(B_n = 0) = 1/2$.



(b)

$$\begin{aligned}\phi_{BB}(m) &= E\left[B_{n+m}B_n\right] = E\left[\left(I_{n+m} + I_{n+m-1}\right)\left(I_n + I_{n-1}\right)\right] \\ &= \phi_{ii}(m) + \phi_{ii}(m-1) + \phi_{ii}(m+1)\end{aligned}$$

Since the sequence $\{I_n\}$ consists of independent symbols :

$$\phi_{ii}(m) = \left\{\begin{array}{cc} E\left[I_{n+m}\right]E\left[I_n\right] = 0 \cdot 0 = 0, & \text{m} \neq 0 \\ E\left[I_n^2\right] = 1, & \text{m} = 0 \end{array}\right\}$$

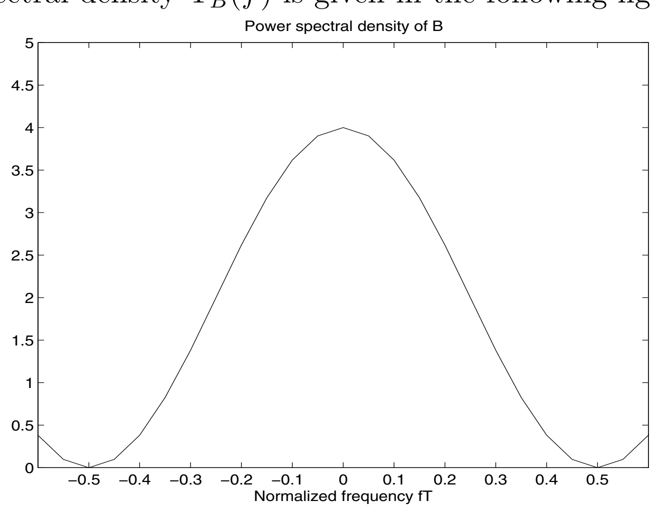
Hence :

$$\phi_{BB}(m) = \left\{\begin{array}{cc} 2, & \text{m} = 0 \\ 1, & \text{m} = \pm 1 \\ 0, & \text{o.w} \end{array}\right\}$$

and

$$\begin{aligned}\Phi_{BB}(f) &= \sum_{m=-\infty}^{\infty} \phi_{BB}(m)\exp(-j2\pi fmT) = 2 + \exp(j2\pi fT) + \exp(-j2\pi fT) \\ &= 2\left[1 + \cos 2\pi fT\right] = 4\cos^2 \pi fT\end{aligned}$$

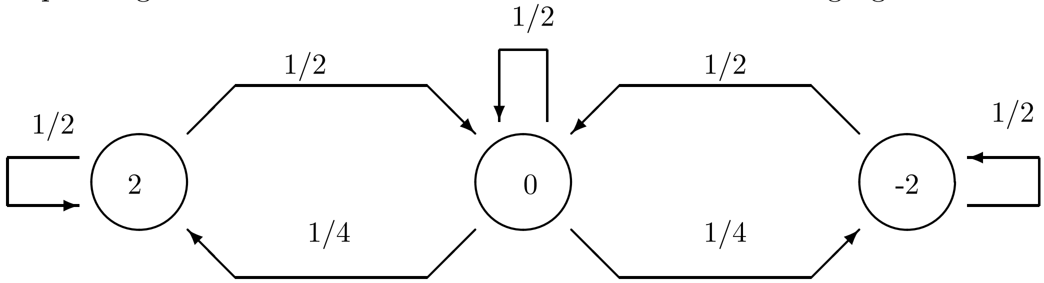
A plot of the power spectral density $\Phi_B(f)$ is given in the following figure :



(c) The transition matrix is :

I_{n-1}	I_n	B_n	I_{n+1}	B_{n+1}
-1	-1	-2	-1	-2
-1	-1	-2	1	0
-1	1	0	-1	0
-1	1	0	1	2
1	-1	0	-1	-2
1	-1	0	1	0
1	1	2	-1	0
1	1	2	1	2

The corresponding Markov chain model is illustrated in the following figure :



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Problem 4.22 :

(a) $I_n = a_n - a_{n-2}$, with the sequence $\{a_n\}$ being uncorrelated random variables (i.e $E(a_{n+m}a_n) = \delta(m)$). Hence :

$$\begin{aligned} \phi_{ii}(m) &= E[I_{n+m}I_n] = E[(a_{n+m} - a_{n+m-2})(a_n - a_{n-2})] \\ &= 2\delta(m) - \delta(m-2) - \delta(m+2) \\ &= \left\{ \begin{array}{cc} 2, & \text{m} = 0 \\ -1, & \text{m} = \pm 2 \\ 0, & \text{o.w.} \end{array} \right\} \end{aligned}$$

(b) $\Phi_{uu}(f) = \frac{1}{T}|G(f)|^2 \Phi_{ii}(f)$ where :

$$\begin{aligned} \Phi_{ii}(f) &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m)\exp(-j2\pi fmT) = 2 - \exp(j4\pi fT) - \exp(-j4\pi fT) \\ &= 2[1 - \cos 4\pi fT] = 4\sin^2 2\pi fT \end{aligned}$$

and

$$|G(f)|^2 = (AT)^2 \left(\frac{\sin \pi fT}{\pi fT}\right)^2$$

Therefore :

$$\Phi_{uu}(f) = 4A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin^2 2\pi fT$$

(c) If $\{a_n\}$ takes the values (0,1) with equal probability then $E(a_n) = 1/2$ and $E(a_{n+m}a_n) = \left\{\begin{array}{ll} 1/4, & \text{m} \neq 0 \\ 1/2, & \text{m} = 0 \end{array}\right\} = [1 + \delta(m)]/4$. Then :

$$\begin{array}{rcl} \phi_{ii}(m) & = & E[I_{n+m}I_n] = 2\phi_{aa}(0) - \phi_{aa}(2) - \phi_{aa}(-2) \\ & = & \frac{1}{4}[2\delta(m) - \delta(m-2) - \delta(m+2)] \end{array}$$

and

$$\begin{array}{l} \Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \exp(-j2\pi fmT) = \sin^2 2\pi fT \\ \Phi_{uu}(f) = A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin^2 2\pi fT \end{array}$$

Thus, we obtain the same result as in (b) , but the magnitude of the various quantities is reduced by a factor of 4 .

Problem 4.23 :

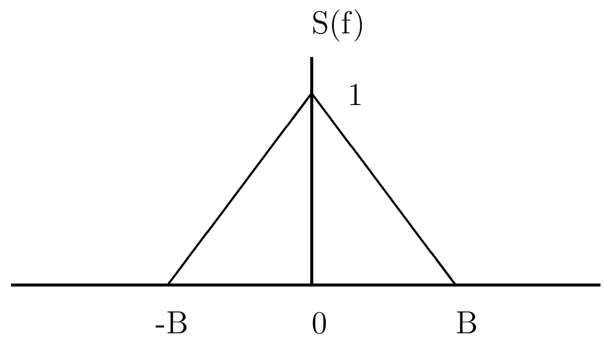
$x(t) = Re\left[u(t) \exp\left(j2\pi f_c t\right)\right]$ where $u(t) = s(t) \pm j\hat{s}(t)$. Hence :

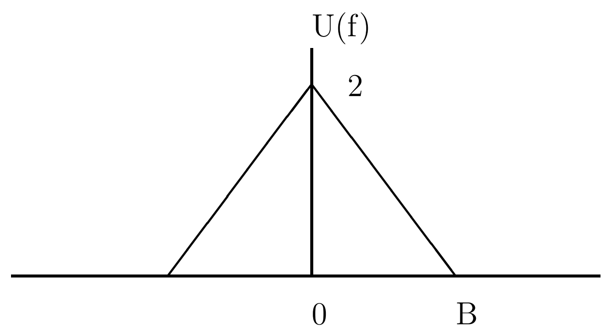
$$U(f) = S(f) \pm j\hat{S}(f) \quad \text{where } \hat{S}(f) = \left\{\begin{array}{ll} -jS(f), & f > 0 \\ jS(f), & f < 0 \end{array}\right\}$$

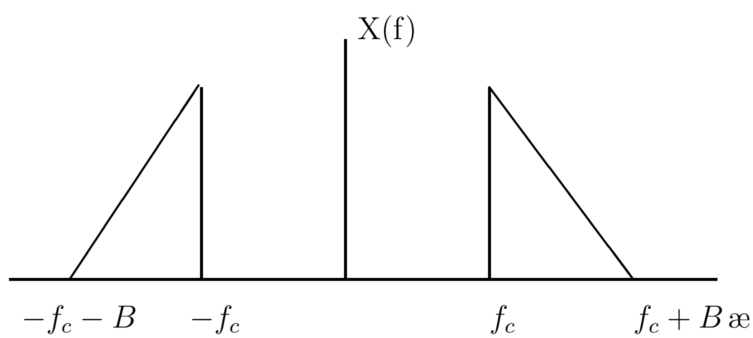
So :

$$U(f) = \left\{\begin{array}{ll} S(f) \pm S(f), & f > 0 \\ S(f) \mp S(f), & f < 0 \end{array}\right\} = \left\{\begin{array}{ll} 2S(f) \text{ or } 0, & f > 0 \\ 0 \text{ or } 2S(f), & f < 0 \end{array}\right\}$$

Since the lowpass equivalent of $x(t)$ is single-sideband, we conclude that $x(t)$ is a single-sideband signal, too. Suppose, for example, that $s(t)$ has the following spectrum. Then, the spectra of the signals $u(t)$ (shown in the figure for the case $u(t) = s(t) + j\hat{s}(t)$) and $x(t)$ are single-sideband







Problem 4.24 :

We may use the result in (4.4.60), where we set $K = 2,\ p_1 = p_2 = 1/2$:

$$\Phi(f) = \frac{1}{T^2} \sum_{l=-\infty}^{\infty} \left| \sum_{i=1}^{2} \frac{1}{2} S_i\left(\frac{l}{T}\right) \right|^2 \delta\left(f - \frac{l}{T}\right) + \frac{1}{T} \sum_{i=1}^{2} \frac{1}{4} \left|S_i(f)\right|^2 - \frac{2}{T} \frac{1}{4} Re\left[S_1(f) S_2^*(f)\right]$$

To simplify the computations we may define the signals over the symmetric interval $-T/2 \leq t \leq T/2$. Then :

$$S_i(f) = \frac{T}{2j} \left[\frac{\sin \pi(f - f_i)T}{\pi(f - f_i)T} - \frac{\sin \pi(f + f_i)T}{\pi(f + f_i)T} \right]$$

(the well-known rectangular pulse spectrum, modulated by $\sin 2\pi f_i t$) and :

$$|S_i(f)|^2 = \left(\frac{T}{2}\right)^2 \left[\left(\frac{\sin\pi(f-f_i)T}{\pi(f-f_i)T}\right)^2 + \left(\frac{\sin\pi(f+f_i)T}{\pi(f+f_i)T}\right)^2\right]$$

where the cross-term involving the product $\frac{\sin\pi(f-f_i)T}{\pi(f-f_i)T} \cdot \frac{\sin\pi(f+f_i)T}{\pi(f+f_i)T}$ is negligible when $f_i >> 0$. Also :

$$\begin{aligned} S_1\left(\tfrac{l}{T}\right) &= \tfrac{T}{2j}\left[\frac{\sin\pi(\frac{l}{T}-\frac{n}{2T})T}{\pi(\frac{l}{T}-\frac{n}{2T})T} - \frac{\sin\pi(\frac{l}{T}+\frac{n}{2T})T}{\pi(\frac{l}{T}+\frac{n}{2T})T}\right] \\ &= \tfrac{T}{2j}\left[\frac{\sin(\pi l-\frac{\pi n}{2})}{(\pi l-\frac{\pi n}{2})} - \frac{\sin(\pi l+\frac{\pi n}{2})}{(\pi l+\frac{\pi n}{2})}\right] \\ &= \tfrac{T}{2j}2l(-1)^{l+1}\left(\sin\tfrac{\pi n}{2}\right)/\pi\left(l^2-n^2/4\right) \\ &= \tfrac{lT}{j}(-1)^{l+1}\frac{\sin\frac{\pi n}{2}}{\pi(l^2-n^2/4)} \end{aligned}$$

and similarly for $S_2(\frac{l}{T})$ (with m instead of n). Note that if $n(m)$ is even then $S_{1(2)}(\frac{l}{T}) = 0$ for all l except at $l = \pm n(m)/2$, where $S_{1(2)}(\frac{n(m)}{2T}) = \pm\frac{T}{2j}$. For this case

$$\frac{1}{T^2}\sum_{l=-\infty}^{\infty}\left|\sum_{i=1}^{2}\frac{1}{2}S_i\left(\frac{l}{T}\right)\right|^2 \delta\left(f-\frac{l}{T}\right) = \frac{1}{16}\left[\delta\left(f-\frac{n}{2T}\right)+\delta\left(f+\frac{n}{2T}\right)+\delta\left(f-\frac{m}{2T}\right)+\delta\left(f-\frac{m}{2T}\right)\right]$$

The third term in (4.4.60) involves the product of $S_1(f)$ and $S_2(f)$ which is negligible since they have little spectral overlap. Hence :

$$\Phi(f) = \frac{1}{16}\left[\delta\left(f-\frac{n}{2T}\right)+\delta\left(f+\frac{n}{2T}\right)+\delta\left(f-\frac{m}{2T}\right)+\delta\left(f-\frac{m}{2T}\right)\right] + \frac{1}{4T}\left[|S_1(f)|^2+|S_2(f)|^2\right]$$

In comparison with the spectrum of the MSK signal, we note that this signal has impulses in the spectrum.

Problem 4.25 :

MFSK signal with waveforms : $s_i(t) = \sin\frac{2\pi i t}{T},\ i = 1, 2, ..., M \quad 0 \le t \le T$
The expression for the power density spectrum is given by (4.4.60) with $K = M$ and $p_i = 1/M$. From Problem 4.23 we have that :

$$S_i(f) = \frac{T}{2j}\left[\frac{\sin\pi(f-f_i)T}{\pi(f-f_i)T} - \frac{\sin\pi(f+f_i)T}{\pi(f+f_i)T}\right]$$

for a signal $s_i(t)$ shifted to the left by $T/2$ (which does not affect the power spectrum). We also have that :

$$S_i\left(\frac{n}{T}\right) = \left\{\begin{array}{rc} \pm T/2j, & n = \pm i \\ 0, & \text{o.w.} \end{array}\right\}$$

Hence from (4.4.60) we obtain :

$$\begin{aligned}\Phi(f) &= \frac{1}{T^2}\left(\frac{1}{M}\right)^2\left(\frac{T^2}{4}\right)\sum_{i=1}^{M}\left[\delta(f-f_i)+\delta(f+f_i)\right] \\ &\quad +\frac{1}{T}\left(\frac{1}{M}\right)^2\sum_{i=1}^{M}\left|S_i(f)\right|^2 \\ &\quad -\frac{2}{T}\sum_{i=1}^{M}\sum_{j=i+1}^{M}\left(\frac{1}{M}\right)^2 Re\left[S_i(f)S_j^*(f)\right] \\ &= \left(\frac{1}{2M}\right)^2\sum_{i=1}^{M}\left[\delta(f-f_i)+\delta(f+f_i)\right]+\frac{1}{TM^2}\sum_{i=1}^{M}\left|S_i(f)\right|^2 \\ &\quad -\frac{2}{TM^2}\sum_{i=1}^{M}\sum_{j=i+1}^{M} Re\left[S_i(f)S_j^*(f)\right]\end{aligned}$$

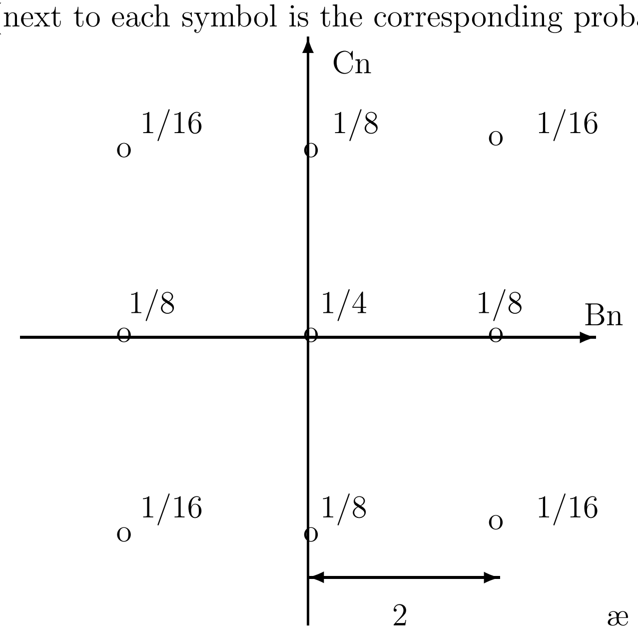
Problem 4.26 :

QPRS signal $v(t)=\sum_n\left(B_n+jC_n\right)u(t-nT),\ B_n=I_n+I_{n-1},\ C_n=J_n+J_{n-1}$.

(a) Similarly to Problem 4.20, the sequence B_n can take the values : $P(B_n=2)=P(B_n=-2)=1/4,\quad P(B_n=0)=1/2$. The same holds for the sequence C_n; since these two sequences are independent :

$$P\left\{B_n=i,C_n=j\right\}=P\left\{B_n=1\right\}P\left\{C_n=j\right\}$$

Hence, since they are also in phase quadrature the signal space representation will be as shown in the following figure (next to each symbol is the corresponding probability of occurrence) :



(b) If we name $Z_n=B_n+jC_n$:

$$\begin{aligned}\phi_{ZZ}(m) &= \tfrac{1}{2}E\left[\left(B_{n+m}+jC_{n+m}\right)\left(B_n-jC_n\right)\right] \\ &= \tfrac{1}{2}\left\{E\left[B_{n+m}B_n\right]+E\left[C_{n+m}C_n\right]\right\}=\tfrac{1}{2}\left(\phi_{BB}(m)+\phi_{CC}(m)\right)=\phi_{BB}(m)=\phi_{CC}(m)\end{aligned}$$

since the sequences B_n, C_n are independent, and have the same statistics. Now, from Problem 4.20 :

$$\phi_{BB}(m) = \left\{ \begin{array}{cc} 2, & \text{m} = 0 \\ 1, & \text{m} = \pm 1 \\ 0, & \text{o.w} \end{array} \right\} = \phi_{CC}(m) = \phi_{ZZ}(m)$$

Hence, from (4-4-11) :

$$\phi_{vs}(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \phi_{BB}(m) \phi_{uu}(\tau - mT) = \phi_{vc}(\tau) = \phi_v(\tau)$$

Also :

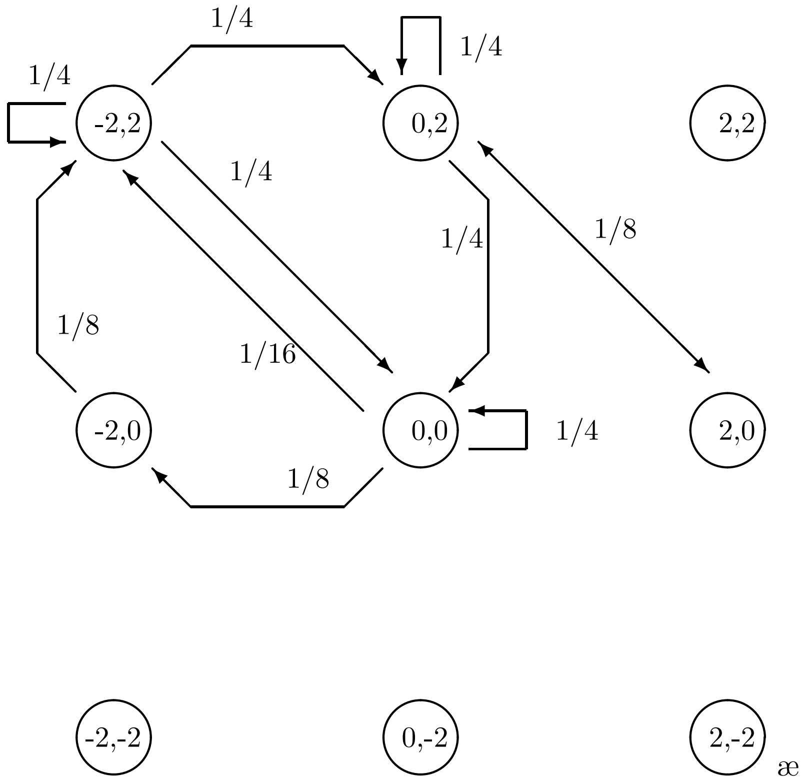
$$\Phi_{vs}(f) = \Phi_{vc}(f) = \Phi_v(f) = \frac{1}{T} |U(f)|^2 \, \Phi_{BB}(f)$$

since the corresponding autocorrelations are the same . From Problem 4.20 : $\Phi_{BB}(f) = 4\cos^2 \pi f T$, so

$$\Phi_{vs}(f) = \Phi_{vc}(f) = \Phi_v(f) = \frac{4}{T} |U(f)|^2 \cos^2 \pi f T$$

Therefore, the composite QPRS signal has the same power density spectrum as the in-phase and quadrature components.

(c) The transition probabilities for the B_n, C_n sequences are independent, so the probability of a transition between one state of the QPRS signal to another state, will be the product of the probabilities of the respective B-transition and C-transition. Hence, the Markov chain model will be the Cartesian product of the Markov model that was derived in Problem 4.20 for the sequence B_n alone. For example, the transition probability from the state $(B_n, C_n) = (0, 0)$ to the same state will be : $P(B_{n+1} = 0 | B_n = 0) \cdot P(C_{n+1} = 0 | C_n = 0) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$ and so on. Below, we give a partial sketch of the Markov chain model; the rest of it can be derived easily, from the symmetries of this model.



Problem 4.27 :

The MSK and offset QPSK signals have the following form :

$$v(t) = \sum_n \left[a_n u(t - 2nT) - j b_n u(t - 2nT - T) \right]$$

where for the QPSK :

$$u(t) = \left\{ \begin{array}{ll} 1, & 0 \leq t \leq 2T \\ 0, & \text{o.w.} \end{array} \right\}$$

and for MSK :

$$u(t) = \left\{ \begin{array}{ll} \sin \frac{\pi t}{2T}, & 0 \leq t \leq 2T \\ 0, & \text{o.w.} \end{array} \right\}$$

The derivation is identical to that given in Sec. 4.4.1 with $2T$ substituted for T. Hence, the result is:

$$\begin{aligned} \phi_{vv}(\tau) &= \tfrac{1}{2T} \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \phi_{uu}(\tau - m2T) \\ &= \tfrac{1}{2T} \sum_{m=-\infty}^{\infty} \left(\sigma_a^2 + \sigma_b^2 \right) \delta(m) \phi_{uu}(\tau - m2T) \\ &= \tfrac{\sigma_a^2}{T} \phi_{uu}(\tau) \end{aligned}$$

and :

$$\Phi_{vv}(f) = \frac{\sigma_a^2}{T} |U(f)|^2$$

For the rectangular pulse of QPSK, we have :

$$\phi_{uu}(\tau) = 2T\left(1 - \frac{|\tau|}{2T}\right), \quad 0 \leq |\tau| \leq 2T$$

For the MSK pulse :

$$\begin{aligned} \phi_{uu}(\tau) &= \int_{-\infty}^{\infty} u(t+\tau)u^*(t)dt = \int_0^{2T-\tau} \sin\frac{\pi t}{2T} \sin\frac{\pi(t+\tau)}{2T}dt \\ &= T\left(1 - \frac{|\tau|}{2T}\right)\cos\frac{\pi|\tau|}{2T} + \frac{\tau}{\pi}\sin\frac{\pi|\tau|}{2T} \end{aligned}$$

Problem 4.28 :

(a) For simplicity we assume binary CPM. Since it is partial response :

$$\begin{aligned} &q(T) = \int_0^T u(t)dt = 1/4 \\ &q(2T) = \int_0^{2T} u(t)dt = 1/2, \qquad q(t) = 1/2, \quad t > 2T \end{aligned}$$

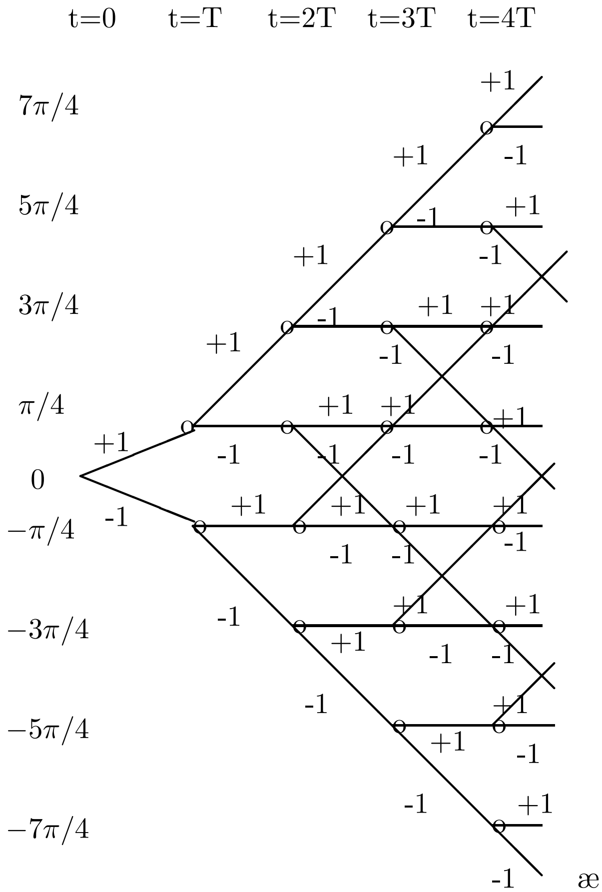
so only the last two symbols will have an effect on the phase :

$$\begin{aligned} \phi(t;\mathbf{I}) &= 2\pi h \sum_{k=-\infty}^{n} I_k q(t-kT), \quad nT \leq t \leq nT+T \\ &= \frac{\pi}{2}\sum_{k=-\infty}^{n-2} I_k + \pi\left(I_{n-1}q(t-(n-1)T) + I_n q(t-nT)\right), \end{aligned}$$

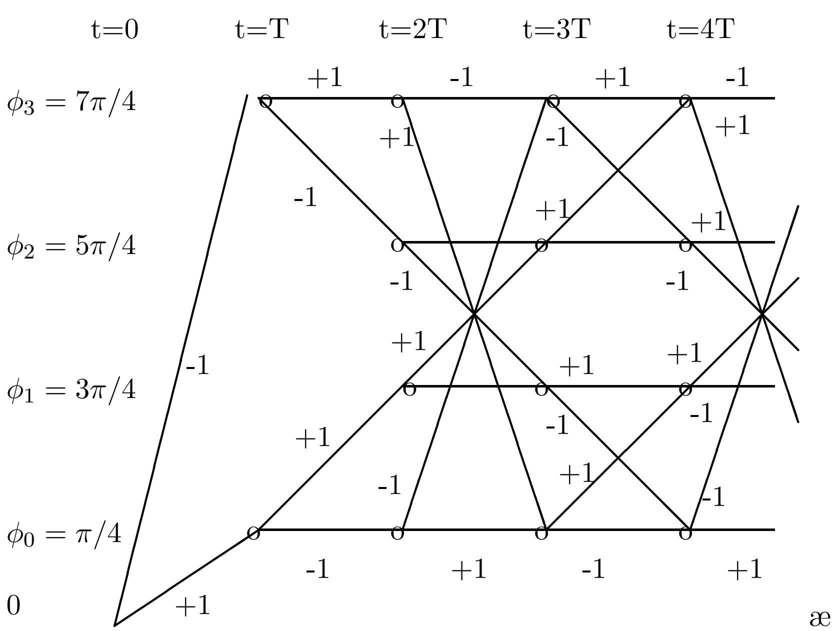
It is easy to see that, after the first symbol, the phase slope is : 0 if I_n, I_{n-1} have different signs, and $sgn(I_n)\pi/(2T)$ if I_n, I_{n-1} have the same sign. At the terminal point $t = (n+1)T$ the phase is :

$$\phi((n+1)T;\mathbf{I}) = \frac{\pi}{2}\sum_{k=-\infty}^{n-1} I_k + \frac{\pi}{4}I_n$$

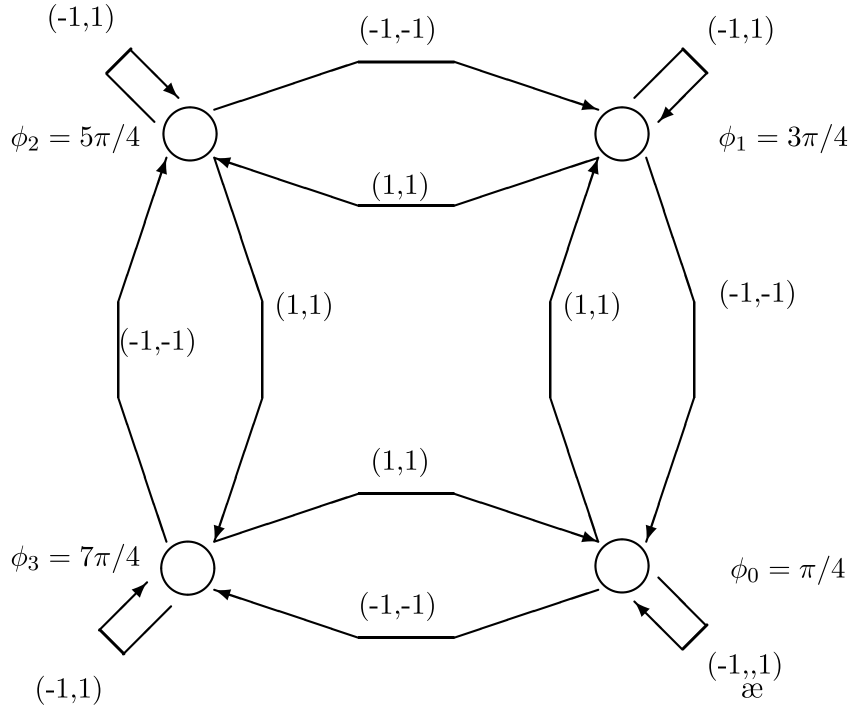
Hence the phase tree is as shown in the following figure :



(b) The state trellis is obtained from the phase-tree modulo 2π:



(c) The state diagram is shown in the following figure (with the (I_n, I_{n-1}) or (I_{n-1}, I_n) that cause the respective transitions shown in parentheses)



Problem 4.29 :

$$\phi(t; \mathbf{I}) = 2\pi h \sum_{k=-\infty}^{n} I_k q(t - kT)$$

(a) Full response binary CPFSK ($q(T) = 1/2$):
(i) $h = 2/3$. At the end of each bit interval the phase is : $2\pi\frac{2}{3}\frac{1}{2}\sum_{k=-\infty}^{n} I_k = \frac{2\pi}{3}\sum_{k=-\infty}^{n} I_k$. Hence the possible terminal phase states are $\{0,\ 2\pi/3,\ 4\pi/3\}$.
(ii) $h = 3/4$. At the end of each bit interval the phase is : $2\pi\frac{3}{4}\frac{1}{2}\sum_{k=-\infty}^{n} I_k = \frac{3\pi}{4}\sum_{k=-\infty}^{n} I_k$. Hence the possible terminal phase states are $\{0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 3\pi/2,\ 7\pi/4\}$

(b) Partial response $L = 3$, binary CPFSK : $q(T) = 1/6,\ q(2T) = 1/3,\ q(3T) = 1/2$. Hence, at the end of each bit interval the phase is :

$$\pi h \sum_{k=-\infty}^{n-2} I_k + 2\pi h \left(I_{n-1}/3 + I_n/6\right) = \pi h \sum_{k=-\infty}^{n-2} I_k + \frac{\pi h}{3}\left(2I_{n-1} + I_n\right)$$

The symbol levels in the parenthesis can take the values $\{-3, -1, 1, 3\}$. So :
(i) $h = 2/3$. The possible terminal phase states are :

$$\{0, 2\pi/9, 4\pi/9, 2\pi/3, 8\pi/9, 10\pi/9, 4\pi/3, 14\pi/9, 16\pi/9\}$$

(ii) $h = 3/4$. The possible terminal phase states are : $\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}$

Problem 4.30 :

The 16-QAM signal is represented as $s(t) = I_n \cos 2\pi f t + Q_n \sin 2\pi f t$, where $I_n = \{\pm 1, \pm 3\}$, $Q_n = \{\pm 1, \pm 3\}$. A superposition of two 4-QAM (4-PSK) signals is :

$$s(t) = G\left[A_n \cos 2\pi f t + B_n \sin 2\pi f t\right] + C_n \cos 2\pi f t + C_n \sin 2\pi f t$$

where $A_n, B_n, C_n, D_n = \{\pm 1\}$. Clearly : $I_n = GA_n + C_n,\ Q_n = GB_n + D_n$. From these equations it is easy to see that $G = 2$ gives the requires equivalence.

Problem 4.31 :

We are given by Equation (4.3-77) that the pulses $c_k(t)$ are defined as

$$c_k(t) = s_0(t) \prod_{n=1}^{L-1} s_0[t + (n + La_{k,n})], \quad 0 \le t \le T \cdot \min_n [L(2 - a_{k,n} - n]$$

Hence, the time support of the pulse $c_k(t)$ is

$$0 \le t \le T \cdot \min_n [L(2 - a_{k,n}) - n]$$

We need to find the index $\hat{n}$ which minimizes $S = L(2 - a_{k,n}) - n$, or equivalently maximizes $S_1 = La_{k,n} + n$:

$$\hat{n} = \arg\max_n [La_{k,n} + n], \quad n = 1, ..., L-1, \quad a_{k,n} = 0, 1$$

It is easy to show that

$$\hat{n} = L - 1 \tag{1}$$

if all $a_{k,n}, \quad n = 0, 1, ..., L-1$ are zero (for a specific k), and

$$\hat{n} = \max\{n : a_{k,n} = 1\} \tag{2}$$

otherwise.
The first case (1) is shown immediately, since if all $a_{k,n}, \quad n = 0, 1, ..., L-1$ are zero, then $\max_n S_1 = \max_n n, \quad n = 0, 1, ..., L-1$. For the second case (2), assume that there are n_1, n_2 such that : $n_1 < n_2$ and $a_{k,n_1} = 1,\ a_{k,n_2} = 0$. Then $S_1(n_1) = L + n_1 > n_2 (= S_1(n_2))$, since $n_2 - n_1 < L - 1$ due to the allowable range of n.
So, finding the binary representation of $k, \quad k = 0, 1, ..., 2^{L-1} - 1$, we find $\hat{n}$ and the corresponding $S(n)$ which gives the extent of the time support of $c_k(t)$:

$$\begin{array}{llllll}
k = 0 & \Rightarrow & a_{k,L-1} = 0, ..., a_{k,2} = 0, \quad a_{k,1} = 0 & \Rightarrow \hat{n} = L-1 & \Rightarrow & S = L+1 \\
k = 1 & \Rightarrow & a_{k,L-1} = 0, ..., a_{k,2} = 0, \quad a_{k,1} = 1 & \Rightarrow \hat{n} = 1 & \Rightarrow & S = L-1 \\
k = 2/3 & \Rightarrow & a_{k,L-1} = 0, ..., a_{k,2} = 1, \quad a_{k,1} = 0/1 & \Rightarrow \hat{n} = 2 & \Rightarrow & S = L-2
\end{array}$$

and so on, using the binary representation of the integers between 1 and $2^{L-1}-1$.

Problem 4.32 :

$s_k(t) = I_k s(t) \Rightarrow S_k(f) = I_k S(f), \quad E(I_k) = \mu_i, \quad \sigma_i^2 = E(I_k^2) - \mu_i^2$

$$\left|\sum_{k=1}^{K} p_k S_k(f)\right|^2 = |S(f)|^2 \left|\sum_{k=1}^{K} p_k I_k\right|^2 = \mu_i^2 \left|S(f)\right|^2$$

Therefore, the discrete frequency component becomes :

$$\frac{\mu_i^2}{T^2} \sum_{n=-\infty}^{\infty} \left|S\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)$$

The continuous frequency component is :

$$\frac{1}{T}\sum_{k=1}^{K} p_k(1-p_k)\left|S_k(f)\right|^2 - \frac{2}{T}\sum\sum_{i<j} p_i p_j Re\left[S_i(f)S_j^*(f)\right]$$

$$= \frac{1}{T}\left|S(f)\right|^2 \left[\sum_{k=1}^{K} p_k \left|I_k\right|^2 - \sum_{k=1}^{K} p_k^2 \left|I_k\right|^2\right] - \frac{2}{T}\sum\sum_{i<j} p_i p_j \left|S(f)\right|^2 \frac{I_i I_j^* + I_i^* I_j}{2}$$

$$= \frac{1}{T}\left|S(f)\right|^2 \left[\sum_{k=1}^{K} p_k \left|I_k\right|^2 - \sum_{k=1}^{K} p_k^2 \left|I_k\right|^2 - \sum\sum_{i<j} p_i p_j \left|S(f)\right| \left({}^2 I_i I_j^* + I_i^* I_j\right)\right]$$

$$= \frac{1}{T}\left|S(f)\right|^2 \left\{\sum_{k=1}^{K} p_k \left|I_k\right|^2 - \left|\sum_{k=1}^{K} p_k I_k\right|^2\right\}$$

$$= \frac{\sigma_i^2}{T}\left|S(f)\right|^2$$

Thus, we have obtained the result in (4.4.18)

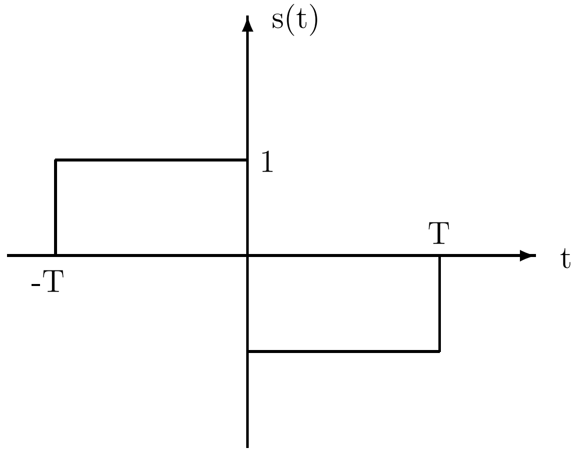
Problem 4.33 :

The line spectrum in (4.4.60) consists of the term :

$$\frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left|\sum_{k=1}^{K} p_k S_k\left(\frac{n}{T}\right)\right|^2 \delta\left(f - \frac{n}{T}\right)$$

Now, if $\sum_{k=1}^{K} p_k s_k(t) = 0$, then $\sum_{k=1}^{K} p_k S_k(f) = 0,\ \forall f$. Therefore, the condition $\sum_{k=1}^{K} p_k s_k(t) = 0$ is sufficient for eliminating the line spectrum.
Now, suppose that $\sum_{k=1}^{K} p_k s_k(t) \neq 0$ for some $t \in [t_0, t_1]$. For example, if $s_k(t) = I_k s(t)$, then $\sum_{k=1}^{K} p_k s_k(t) = s(t) \sum_{k=1}^{K} p_k I_k$, where $\sum_{k=1}^{K} p_k I_k \equiv \mu_i \neq 0$ and $s(t)$ is a signal pulse. Then, the

line spectrum vanishes if $S(n/T) = 0$ for all n. A signal pulse that satisfies this condition is shown below :



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In this case, $S(f) = T\left(\frac{\sin \pi T f}{\pi T f}\right) \sin \pi T f$, so that $S(n/T) = 0$ for all n. Therefore, the condition $\sum_{k=1}^{K} p_k s_k(t) = 0$ is not necessary.

Problem 4.34 :

(a) Since : $\mu_a = 0,\ \sigma_a^2 = 1$, we have : $\Phi_{ss}(f) = \frac{1}{T} |G(f)|^2$. But :

$$\begin{aligned} G(f) &= \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j2\pi f T/4} - \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j2\pi f 3T/4} \\ &= \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j\pi f T} (2j \sin \pi f T/2) \\ &= jT \frac{\sin^2 \pi f T/2}{\pi f T/2} e^{-j\pi f T} \Rightarrow \\ |G(f)|^2 &= T^2 \left(\frac{\sin^2 \pi f T/2}{\pi f T/2}\right)^2 \Rightarrow \\ \Phi_{ss}(f) &= T \left(\frac{\sin^2 \pi f T/2}{\pi f T/2}\right)^2 \end{aligned}$$

(b) For non-independent information sequence the power spectrum of $s(t)$ is given by : $\Phi_{ss}(f) =$

$\frac{1}{T}|G(f)|^2 \Phi_{bb}(f)$. But :

$$\begin{aligned}\phi_{bb}(m) &= E\left[b_{n+m}b_n\right] \\ &= E\left[a_{n+m}a_n\right] + kE\left[a_{n+m-1}a_n\right] + kE\left[a_{n+m}a_{n-1}\right] + k^2E\left[a_{n+m-1}a_{n-1}\right] \\ &= \left\{\begin{array}{rr} 1+k^2, & \mathrm{m}=0 \\ k, & \mathrm{m}=\pm 1 \\ 0, & \mathrm{o.w.}\end{array}\right\}\end{aligned}$$

Hence :

$$\Phi_{bb}(f) = \sum_{m=-\infty}^{\infty} \phi_{bb}(m)e^{-j2\pi fmT} = 1 + k^2 + 2k\cos 2\pi fT$$

We want :

$$\Phi_{ss}(1/T) = 0 \Rightarrow \Phi_{bb}(1/T) = 0 \Rightarrow 1 + k^2 + 2k = 0 \Rightarrow k = -1$$

and the resulting power spectrum is :

$$\Phi_{ss}(f) = 4T\left(\frac{\sin^2\pi fT/2}{\pi fT/2}\right)^2 \sin^2\pi fT$$

(c) The requirement for zeros at $f = l/4T,\ l = \pm 1, \pm 2, ...$ means : $\Phi_{bb}(l/4T) = 0 \Rightarrow 1 + k^2 + 2k\cos\pi l/2 = 0$, which cannot be satisfied for all l. We can avoid that by using precoding in the form :$b_n = a_n + ka_{n-4}$. Then :

$$\phi_{bb}(m) = \left\{\begin{array}{rr} 1+k^2, & \mathrm{m}=0 \\ k, & \mathrm{m}=\pm 4 \\ 0, & \mathrm{o.w.}\end{array}\right\} \Rightarrow \Phi_{bb}(f) = 1 + k^2 + 2k\cos 2\pi f4T$$

and , similarly to (b), a value of $k = -1$, will zero this spectrum in all multiples of $1/4T$.

Problem 4.35 :

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}, \quad \rho = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

By straightforward matrix multiplication we verify that indeed :

$$\mathbf{P}^4\rho = -\frac{1}{4}\rho$$

Problem 4.36 :

(a) The power spectral density of the FSK signal may be evaluated by using equation (4-4-60) with $K = 2$ (binary) signals and probabilities $p_0 = p_1 = \frac{1}{2}$. Thus, when the condition that the carrier phase θ_0 and and θ_1 are fixed, we obtain

$$\Phi(f) = \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} |S_0(\frac{n}{T}) + S_1(\frac{n}{T})|^2 \delta(f - \frac{n}{T}) + \frac{1}{4T}|S_0(f) - S_1(f)|^2$$

where $S_0(f)$ and $S_1(f)$ are the fourier transforms of $s_0(t)$ and $s_1(t)$. In particular :

$$\begin{aligned} S_0(f) &= \int_0^T s_0(t) e^{-j2\pi f t}\, dt \\ &= \sqrt{\frac{2\mathcal{E}_b}{T}} \int_0^T \cos(2\pi f_0 t + \theta_0) e^{j2\pi f t}\, dt, \qquad f_0 = f_c - \frac{\Delta f}{2} \\ &= \sqrt{\frac{T\mathcal{E}_b}{2}} \left[\frac{\sin \pi T(f - f_0)}{\pi(f - f_0)} + \frac{\sin \pi T(f + f_0)}{\pi(f + f_0)} \right] e^{-j\pi f T} e^{j\theta_0} \end{aligned}$$

Similarly :

$$\begin{aligned} S_1(f) &= \int_0^T s_1(t) e^{-j2\pi f t}\, dt \\ &= \sqrt{\frac{T\mathcal{E}_b}{2}} \left[\frac{\sin \pi T(f - f_1)}{\pi(f - f_1)} + \frac{\sin \pi T(f + f_1)}{\pi(f + f_1)} \right] e^{-j\pi f T} e^{j\theta_1} \end{aligned}$$

where $f_1 = f_c + \frac{\Delta f}{2}$. By expressing $\Phi(f)$ as :

$$\begin{aligned} \Phi(f) &= \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[|S_0(\frac{n}{T})|^2 + |S_1(\frac{n}{T})|^2 + 2\text{Re}[S_0(\frac{n}{T}) S_1^*(\frac{n}{T})] \right] \delta(f - \frac{n}{T}) \\ &+ \frac{1}{4T} \left[|S_0(f)|^2 + |S_1(f)|^2 - 2\text{Re}[S_0(f) S_1^*(f)] \right] \end{aligned}$$

we note that the carrier phases θ_0 and θ_1 affect only the terms $\text{Re}(S_0 S_1^*)$. If we average over the random phases, these terms drop out. Hence, we have :

$$\begin{aligned} \Phi(f) &= \frac{1}{4T^2} \sum_{n=-\infty}^{\infty} \left[|S_0(\frac{n}{T})|^2 + |S_1(\frac{n}{T})|^2 \right] \delta(f - \frac{n}{T}) \\ &+ \frac{1}{4T} \left[|S_0(f)|^2 + |S_1(f)|^2 \right] \end{aligned}$$

where :

$$|S_k(f)|^2 = \frac{T\mathcal{E}_b}{2} \left| \frac{\sin \pi T(f - f_k)}{\pi(f - f_k)} + \frac{\sin \pi T(f + f_k)}{\pi(f + f_k)} \right|^2, \quad k = 0, 1$$

Note that the first term in $\Phi(f)$ consists of a sequence of samples and the second term constitutes the continuous spectrum.

(b) Note that :

$$|S_k(f)|^2 = \frac{T\mathcal{E}_b}{2}\left[\left(\frac{\sin \pi T(f-f_k)}{\pi(f-f_k)}\right)^2 + \left(\frac{\sin \pi T(f+f_k)}{\pi(f+f_k)}\right)^2\right]$$

because the product

$$\frac{\sin \pi T(f-f_k)}{\pi(f-f_k)} \times \frac{\sin \pi T(f+f_k)}{\pi(f+f_k)} \approx 0$$

if f_k is large enough. Hence $|S_k(f)|^2$ decays proportionally to $\frac{1}{(f-f_k)^2}$ *approx*$\frac{1}{f^2}$ for $f \gg f_c$. Consequently, $\Phi(f)$ exhibits the same behaviour.

CHAPTER 5

Problem 5.1 :

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

where $sgn(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

(b) The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 5.2 :

(a) The impulse response of the matched filter is :

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

(b) The output of the matched filter at $t = T$ is :

$$\begin{aligned} g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\ &\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\ &= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\ &= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right] \end{aligned}$$

(c) The output of the correlator at $t = T$ is :

$$\begin{aligned} q(T) &= \int_0^T s^2(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T \tau^2 \cos^2(2\pi f_c \tau)d\tau \end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

Problem 5.3 :

(a) In binary DPSK, the information bit 1 is transmitted by shifting the phase of the carrier by π radians relative to the phase in the previous interval, while if the information bit is 0 then the phase is not shifted. With this in mind :

Data :	1	1	0	1	0	0	0	1	0	1	1	0
Phase θ : (π)	0	π	π	0	0	0	0	π	π	0	π	π

Note : since the phase in the first bit interval is 0, we conclude that the phase before that was π.

(b) We know that the power spectrum of the equivalent lowpass signal $u(t)$ is :

$$\Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2 \Phi_{ii}(f)$$

where $G(f) = AT\frac{\sin \pi f T}{\pi f T}$, is the spectrum of the rectangular pulse of amplitude A that is used, and $\Phi_{ii}(f)$ is the power spectral density of the information sequence. It is straightforward to see that the information sequence I_n is simply the phase of the lowpass signal, i.e. it is $e^{j\pi}$ or e^{j0} depending on the bit to be transmitted $a_n(= 0, 1)$. We have :

$$I_n = e^{j\theta_n} = e^{j\pi a_n} e^{j\theta_{n-1}} = e^{j\pi \sum_k a_k}$$

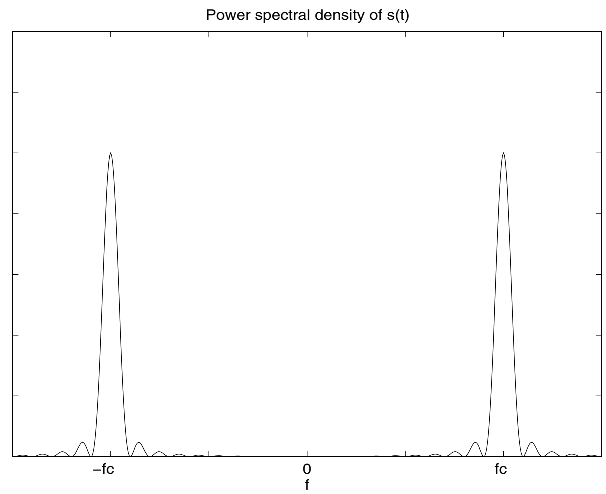
The statistics of I_n are (remember that $\{a_n\}$ are uncorrelated) :

$$\begin{aligned} E\left[I_n\right] &= E\left[e^{j\pi\sum_k a_k}\right] = \prod_k E\left[e^{j\pi a_k}\right] = \prod_k \left[\tfrac{1}{2}e^{j\pi} - \tfrac{1}{2}e^{j0}\right] = \prod_k 0 = 0 \\ E\left[\left|I_n\right|^2\right] &= E\left[e^{j\pi\sum_k a_k} e^{-j\pi\sum_k a_k}\right] = 1 \\ E\left[I_{n+m}I_n^*\right] &= E\left[e^{j\pi\sum_{k=0}^{n+m} a_k} e^{-j\pi\sum_{k=0}^{n} a_k}\right] = E\left[e^{j\pi\sum_{k=n+1}^{m} a_k}\right] = \prod_{k=n+1}^{m} E\left[e^{j\pi a_k}\right] = 0 \end{aligned}$$

Hence, I_n is an uncorrelated sequence with zero mean and unit variance, so $\Phi_{ii}(f) = 1$, and

$$\begin{aligned} \Phi_{uu}(f) &= \tfrac{1}{T}\left|G(f)\right|^2 = A^2 T\frac{\sin \pi f T}{\pi f T} \\ \Phi_{ss}(f) &= \tfrac{1}{2}\left[\Phi_{uu}(f - f_c) + \Phi_{uu}(-f - f_c)\right] \end{aligned}$$

A sketch of the signal power spectrum $\Phi_{ss}(f)$ is given in the following figure :



Problem 5.4 :

(a) The correlation type demodulator employes a filter :

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{T}} & 0 \leq \text{t} \leq \text{T} \\ 0 & \text{o.w} \end{array} \right\}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance :

$$\sigma_n^2 \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\text{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \text{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\mathrm{PM}(\mathbf{r},\mathbf{s}_0)}{\mathrm{PM}(\mathbf{r},\mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2-r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently :

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}$$

where

$$\mathrm{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 5.5 :

Since $\{f_n(t)\}$ constitute an orthonormal basis for the signal space : $r(t) = \sum_{n=1}^{N} r_n f_n(t),\ s_m(t) =$

$\sum_{n=1}^{N} s_{mn} f_n(t)$. Hence, for any m :

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= 2\int_0^T r(t) s_m(t) dt - \int_0^T s_m^2(t) dt \\ &= 2\int_0^T \sum_{n=1}^{N} r_n f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt - \int_0^T \sum_{n=1}^{N} s_{mn} f_n(t) \sum_{l=1}^{N} s_{ml} f_l(t) dt \\ &= 2\sum_{n=1}^{N} r_n \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt - \sum_{n=1}^{N} s_{mn} \sum_{l=1}^{N} s_{ml} \int_0^T f_n(t) f_l(t) dt \\ &= 2\sum_{n=1}^{N} r_n s_{mn} - \sum_{n=1}^{N} s_{mn}^2 \end{aligned}$$

where we have exploited the orthonormality of $\{f_n(t)\}$: $\int_0^T f_n(t) f_l(t) dt = \delta_{nl}$. The last form is indeed the original form of the correlation metrics $C(\mathbf{r}, \mathbf{s}_m)$.

Problem 5.6 :

The SNR at the filter output will be :

$$SNR = \frac{|y(T)|^2}{E\left[|n(T)|^2\right]}$$

where $y(t)$ is the part of the filter output that is due to the signal $s_l(t)$, and $n(t)$ is the part due to the noise $z(t)$. The denominator is :

$$\begin{aligned} E\left[|n(T)|^2\right] &= \int_0^T \int_0^T E\left[z(a) z^*(b)\right] h_l(T-a) h_l^*(T-b) da db \\ &= 2N_0 \int_0^T |h_l(T-t)|^2 dt \end{aligned}$$

so we want to maximize :

$$SNR = \frac{\left|\int_0^T s_l(t) h_l(T-t) dt\right|^2}{2N_0 \int_0^T |h_l(T-t)|^2 dt}$$

From Schwartz inequality :

$$\left|\int_0^T s_l(t) h_l(T-t) dt\right|^2 \leq \int_0^T |h_l(T-t)|^2 dt \int_0^T |s_l(t)|^2 dt$$

Hence :

$$SNR \leq \frac{1}{2N_0} \int_0^T |s_l(t)|^2 dt = \frac{\mathcal{E}}{N_0} = SNR_{\max}$$

and the maximum occurs when :

$$s_l(t) = h_l^*(T-t) \Leftrightarrow h_l(t) = s_l^*(T-t)$$

Problem 5.7 :

$$N_{mr} = Re\left[\int_0^T z(t) f_m^*(t) dt\right]$$

(a) Define $a_m = \int_0^T z(t) f_m^*(t) dt$. Then, $N_{mr} = Re(a_m) = \frac{1}{2}\left[a_m + a_m^*\right]$.

$$E(N_{mr}) = Re\left[\int_0^T E\left(z(t)\right) f_m^*(t) dt\right] = 0$$

since, $E\left[z(t)\right] = 0$. Also :

$$E\left(N_{mr}^2\right) = E\left[\frac{a_m^2 + \left(a_m^*\right)^2 + 2a_m a_m^*}{4}\right]$$

But $E\left(a_m^2\right) = E\left[\int_0^T \int_0^T z(a) z(b) f_m^*(a) f_m^*(b) da db\right] = 0$, since $E\left[z(a)z(b)\right] = 0$ (Problem 4.3), and the same is true for $E\left[\left(a_m^*\right)^2\right] = 0$, since $E\left[z^*(a) z^*(b)\right] = 0$ Hence :

$$\begin{array}{rcl} E\left(N_{mr}^2\right) & = & E\left[\frac{a_m a_m^*}{2}\right] = \frac{1}{2}\int_0^T \int_0^T E\left[z(a) z^*(b)\right] f_m^*(a) f_m(b) da db \\ & = & N_0 \int_0^T \left|f_m(a)\right|^2 da = 2\mathcal{E} N_0 \end{array}$$

(b) For $m \neq k$:

$$\begin{array}{rcl} E\left[N_{mr} N_{kr}\right] & = & E\left[\frac{a_m + a_m^*}{2} \frac{a_k + a_k^*}{2}\right] \\ & = & E\left[\frac{a_m a_k + a_m^* a_k + a_m a_k^* + a_m^* a_k^*}{4}\right] \end{array}$$

But, similarly to part (a), $E\left[a_m a_k\right] = E\left[a_m^* a_k^*\right] = 0$, hence, $E\left[N_{mr} N_{kr}\right] = E\left[\frac{a_m^* a_k + a_m a_k^*}{4}\right]$. Now :

$$\begin{array}{rcl} E\left[a_m a_k^*\right] & = & \int_0^T \int_0^T E\left[z(a) z^*(b)\right] f_m^*(a) f_k(b) da db \\ & = & 2N_0 \int_0^T f_m^*(a) f_k(a) da = 0 \end{array}$$

since, for $m \neq k$, the waveforms are orthogonal.
Similarly : $E\left[a_m^* a_k\right] = 0$, hence : $E\left[N_{mr} N_{kr}\right] = 0$.

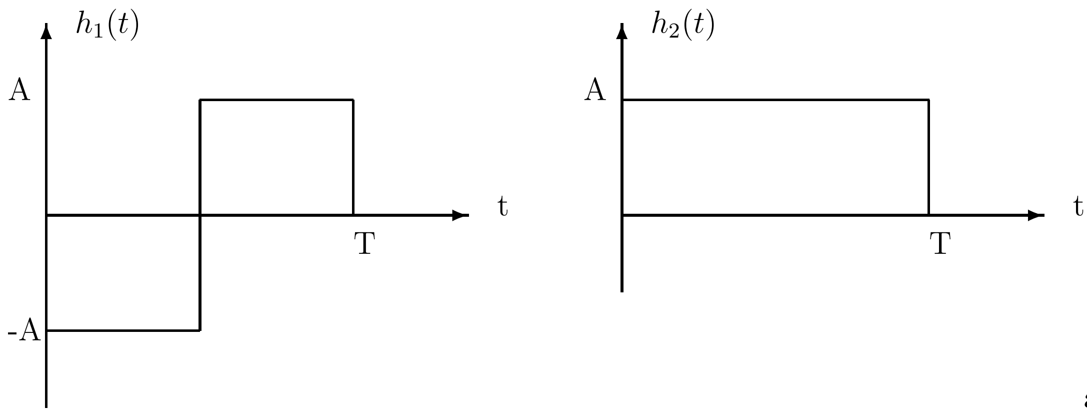
Problem 5.8 :

(a) Since the given waveforms are the equivalent lowpass signals :

$$\begin{array}{l} \mathcal{E}_1 = \frac{1}{2}\int_0^T \left|s_1(t)\right|^2 dt = \frac{1}{2}A^2 \int_0^T dt = A^2 T/2 \\ \mathcal{E}_2 = \frac{1}{2}\int_0^T \left|s_2(t)\right|^2 dt = \frac{1}{2}A^2 \int_0^T dt = A^2 T/2 \end{array}$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}} \int_0^T s_1(t) s_2^*(t) dt = 0$.

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T - t)$. The following figure shows $h_i(t)$:



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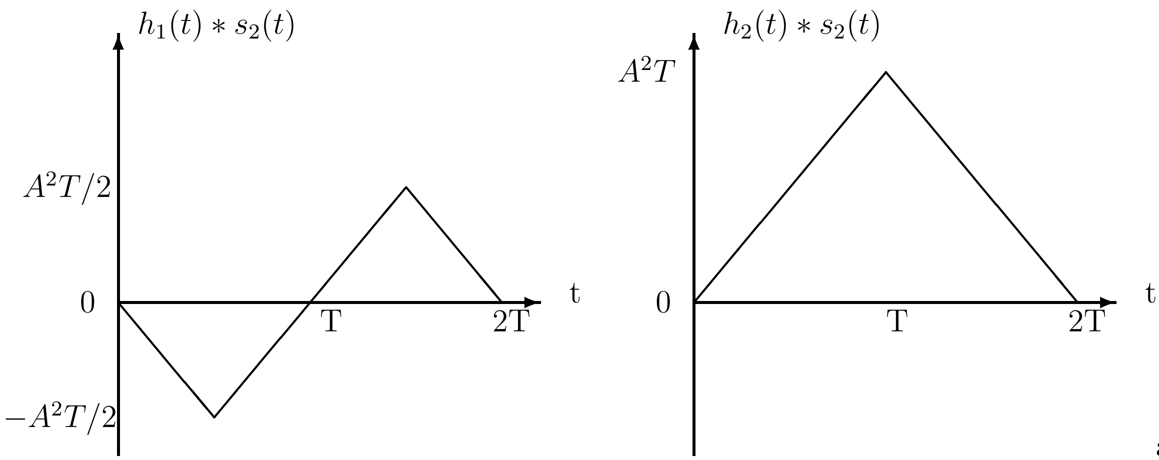
(c)



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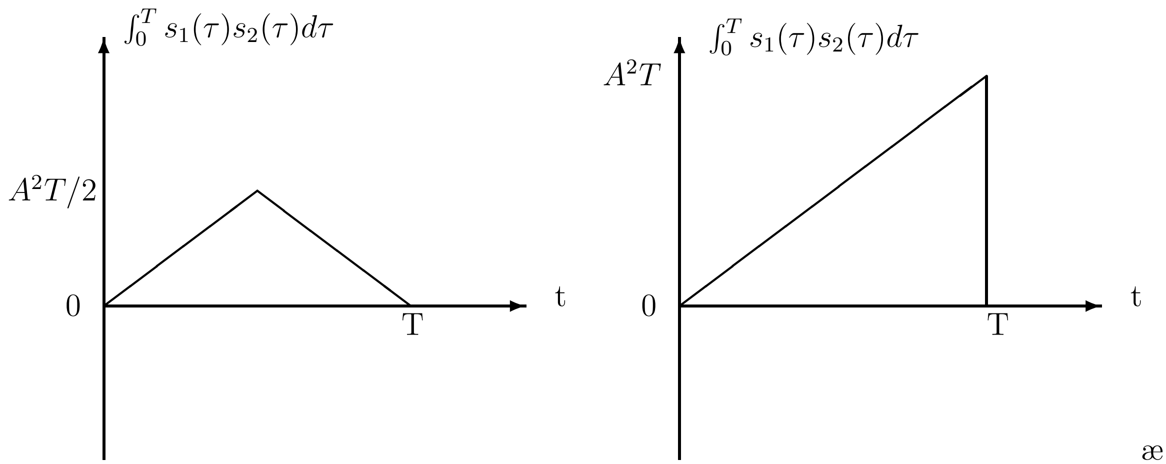
(d)



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(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

Problem 5.9 :

(a) The joint pdf of a, b is :

$$p_{ab}(a,b) = p_{xy}(a - m_r, b - m_i) = p_x(a - m_r)p_y(b - m_i) = \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left[(a-m_r)^2+(b-m_i)^2\right]}$$

(b) $u = \sqrt{a^2 + b^2}$, $\quad \phi = \tan^{-1} b/a \Rightarrow a = u\cos\phi,\ b = u\sin\phi$ The Jacobian of the transformation is : $J(a,b) = \begin{vmatrix} \partial a/\partial u & \partial a/\partial\phi \\ \partial b/\partial u & \partial b/\partial\phi \end{vmatrix} = u$, hence :

$$\begin{aligned} p_{u\phi}(u,\phi) &= \frac{u}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left[(u\cos\phi-m_r)^2+(u\sin\phi-m_i)^2\right]} \\ &= \frac{u}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left[u^2+M^2-2uM\cos(\phi-\theta)\right]} \end{aligned}$$

where we have used the transformation :

$$\left\{ \begin{array}{l} M = \sqrt{m_r^2 + m_i^2} \\ \theta = \tan^{-1} m_i/m_r \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} m_r = M\cos\theta \\ m_i = M\sin\theta \end{array} \right\}$$

(c)

$$\begin{aligned} p_u(u) &= \int_0^{2\pi} p_{u\phi}(u,\phi)d\phi \\ &= \frac{u}{2\pi\sigma^2}e^{-\frac{u^2+M^2}{2\sigma^2}}\int_0^{2\pi} e^{-\frac{1}{2\sigma^2}[-2uM\cos(\phi-\theta)]}d\phi \\ &= \frac{u}{\sigma^2}e^{-\frac{u^2+M^2}{2\sigma^2}}\frac{1}{2\pi}\int_0^{2\pi} e^{uM\cos(\phi-\theta)/\sigma^2}d\phi \\ &= \frac{u}{\sigma^2}e^{-\frac{u^2+M^2}{2\sigma^2}}I_o\left(uM/\sigma^2\right) \end{aligned}$$

Problem 5.10 :

(a) $U = Re\left[\int_0^T r(t)s^*(t)dt\right]$, where $r(t) = \left\{\begin{array}{c} s(t)+z(t) \\ -s(t)+z(t) \\ z(t) \end{array}\right\}$ depending on which signal was sent. If we assume that $s(t)$ was sent :

$$U = Re\left[\int_0^T s(t)s^*(t)dt\right] + Re\left[\int_0^T z(t)s^*(t)dt\right] = 2E + N$$

where $E = \frac{1}{2}\int_0^T s(t)s^*(t)dt$, and $N = Re\left[\int_0^T z(t)s^*(t)dt\right]$ is a Gaussian random variable with zero mean and variance $2EN_0$ (as we have seen in Problem 5.7). Hence, given that $s(t)$ was sent, the probability of error is :

$$P_{e1} = P(2E+N<A) = P\left(N < -(2E-A)\right) = Q\left(\frac{2E-A}{\sqrt{2N_0E}}\right)$$

When $-s(t)$ is transmitted : $U = -2E+N$, and the corresponding conditional error probability is :

$$P_{e2} = P(-2E+N>-A) = P\left(N > (2E-A)\right) = Q\left(\frac{2E-A}{\sqrt{2N_0E}}\right)$$

and finally, when 0 is transmitted : $U = N$, and the corresponding error probability is :

$$P_{e3} = P(N>A \text{ or } N<-A) = 2P(N>A) = 2Q\left(\frac{A}{\sqrt{2N_0E}}\right)$$

(b)

$$P_e = \frac{1}{3}\left(P_{e1}+P_{e2}+P_{e3}\right) = \frac{2}{3}\left[Q\left(\frac{2E-A}{\sqrt{2N_0E}}\right) + Q\left(\frac{A}{\sqrt{2N_0E}}\right)\right]$$

(c) In order to minimize P_e :

$$\frac{dP_e}{dA} = 0 \Rightarrow A = E$$

where we differentiate $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt$ with respect to x, using the Leibnitz rule : $\frac{d}{dx}\left(\int_{f(x)}^\infty g(a)da\right) = -\frac{df}{dx}g(f(x))$. Using this threshold :

$$P_e = \frac{4}{3}Q\left(\frac{E}{\sqrt{2N_0E}}\right) = \frac{4}{3}Q\left(\sqrt{\frac{E}{2N_0}}\right)$$

Problem 5.11 :

(a) The transmitted energy is :

$$\begin{array}{rcl} \mathcal{E}_1 & = & \frac{1}{2}\int_0^T |s_1(t)|^2 \, dt = A^2T/2 \\ \mathcal{E}_2 & = & \frac{1}{2}\int_0^T |s_2(t)|^2 \, dt = A^2T/2 \end{array}$$

(b) The correlation coefficient for the two signals is :

$$\rho = \frac{1}{2\mathcal{E}} \int_0^T s_1(t)s_2^*(t)dt = 1/2$$

Hence, the bit error probability for coherent detection is :

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}(1-\rho)}\right) = Q\left(\sqrt{\frac{\mathcal{E}}{2N_0}}\right)$$

(c) The bit error probability for non-coherent detection is given by (5-4-53) :

$$P_{2,nc} = Q_1(a,b) - \frac{1}{2}e^{-\left(a^2+b^2\right)/2}I_0(ab)$$

where $Q_1(.)$ is the generalized Marcum Q function (given in (2-1-123)) and :

$$\begin{array}{l} a = \sqrt{\frac{\mathcal{E}}{2N_0}\left(1-\sqrt{1-|\rho|^2}\right)} = \sqrt{\frac{\mathcal{E}}{2N_0}\left(1-\frac{\sqrt{3}}{2}\right)} \\ b = \sqrt{\frac{\mathcal{E}}{2N_0}\left(1+\sqrt{1-|\rho|^2}\right)} = \sqrt{\frac{\mathcal{E}}{2N_0}\left(1+\frac{\sqrt{3}}{2}\right)} \end{array}$$

Problem 5.12 :

The correlation of the two signals in binary FSK is:

$$\rho = \frac{\sin(2\pi \Delta f T)}{2\pi \Delta f T}$$

To find the minimum value of the correlation, we set the derivative of ρ with respect to Δf equal to zero. Thus:

$$\frac{\vartheta \rho}{\vartheta \Delta f} = 0 = \frac{\cos(2\pi \Delta f T) 2\pi T}{2\pi \Delta f T} - \frac{\sin(2\pi \Delta f T)}{(2\pi \Delta f T)^2} 2\pi T$$

and therefore :

$$2\pi \Delta f T = \tan(2\pi \Delta f T)$$

Solving numerically (or graphically) the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$2\pi \Delta f T = 4.4934 \Longrightarrow \Delta f = \frac{0.7151}{T}$$

and the value of ρ is -0.2172.
We know that the probability of error can be expressed in terms of the distance d_{12} between the signal points, as :

$$P_e = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]$$

where the distance between the two signal points is :

$$d_{12}^2 = 2\mathcal{E}_b(1 - \rho)$$

and therefore :

$$P_e = Q\left[\sqrt{\frac{2\mathcal{E}_b(1-\rho)}{2N_0}}\right] = Q\left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}}\right]$$

Problem 5.13 :

(a) It is straightforward to see that :

$$\begin{array}{rl} \text{Set I} & : \text{Four} - \text{level PAM} \\ \text{Set II} & : \text{Orthogonal} \\ \text{Set III} & : \text{Biorthogonal} \end{array}$$

(b) The transmitted waveforms in the first set have energy : $\frac{1}{2}A^2$ or $\frac{1}{2}9A^2$. Hence for the first set the average energy is :

$$\mathcal{E}_1 = \frac{1}{4}\left(2\frac{1}{2}A^2 + 2\frac{1}{2}9A^2\right) = 2.5A^2$$

All the waveforms in the second and third sets have the same energy : $\frac{1}{2}A^2$.Hence :

$$\mathcal{E}_2 = \mathcal{E}_3 = A^2/2$$

(c) The average probability of a symbol error for M-PAM is (5-2-45) :

$$P_{4,PAM} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{A^2}{N_0}}\right)$$

(d) For coherent detection, a union bound can be given by (5-2-25) :

$$P_{4,orth} < (M-1)\, Q\left(\sqrt{\mathcal{E}_s/N_0}\right) = 3Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$$

while for non-coherent detection :

$$P_{4,orth,nc} \leq (M-1)\, P_{2,nc} = 3\frac{1}{2}e^{-\mathcal{E}_s/2N_0} = \frac{3}{2}e^{-A^2/4N_0} \qquad ??$$

(e) It is not possible to use non-coherent detection for a biorthogonal signal set : e.g. without phase knowledge, we cannot distinguish between the signals $u_1(t)$ and $u_3(t)$ (or $u_2(t)/u_4(t)$).

(f) The bit rate to bandwidth ratio for M-PAM is given by (5-2-85) :

$$\left(\frac{R}{W}\right)_1 = 2\log_2 M = 2\log_2 4 = 4$$

For orthogonal signals we can use the expression given by (5-2-86) or notice that we use a symbol interval 4 times larger than the one used in set I, resulting in a bit rate 4 times smaller :

$$\left(\frac{R}{W}\right)_2 = \frac{2\log_2 M}{M} = 1$$

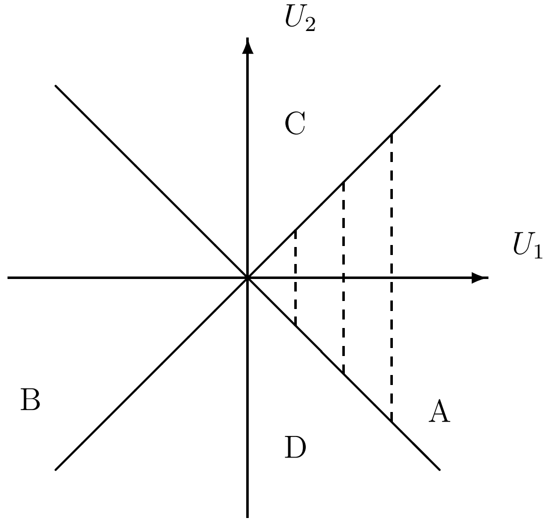
Finally, the biorthogonal set has double the bandwidth efficiency of the orthogonal set :

$$\left(\frac{R}{W}\right)_3 = 2$$

Hence, set I is the most bandwidth efficient (at the expense of larger average power), but set III will also be satisfactory.

Problem 5.14 :

The following graph shows the decision regions for the four signals :

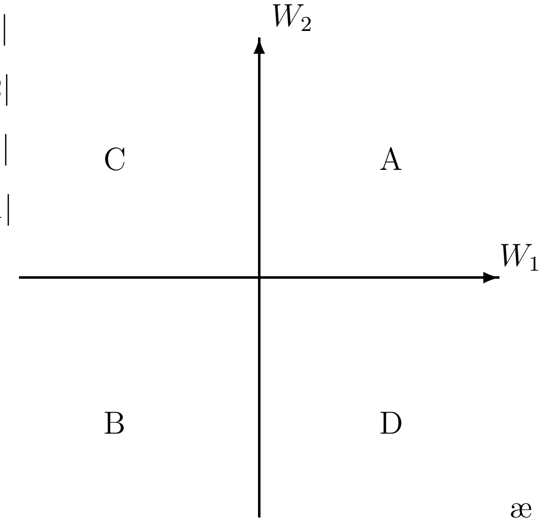


$A = U1 > +|U2|$

$B = U1 < -|U2|$

$C = U2 > +|U1|$

$D = U2 < -|U1|$



As we see, using the transformation $W_1 = U_1 + U_2,\ W_2 = U_1 - U_2$ alters the decision regions to : $(W_1 > 0, W_2 > 0 \rightarrow s_1(t); W_1 > 0, W_2 < 0 \rightarrow s_2(t);\ \text{etc.})$. Assuming that $s_1(t)$ was transmitted, the outputs of the matched filters will be :

$$\begin{array}{l} U_1 = 2\mathcal{E} + N_{1r} \\ U_2 = N_{2r} \end{array}$$

where N_{1r}, N_{2r} are uncorrelated (Prob. 5.7) Gaussian-distributed terms with zero mean and variance $2\mathcal{E}N_0$. Then :

$$\begin{array}{l} W_1 = 2\mathcal{E} + (N_{1r} + N_{2r}) \\ W_2 = 2\mathcal{E} + (N_{1r} - N_{2r}) \end{array}$$

will be Gaussian distributed with means : $E[W_1] = E[W_2] = 2\mathcal{E}$, and variances : $E[W_1^2] = E[W_2^2] = 4\mathcal{E}N_0$. Since U_1, U_2 are independent, it is straightforward to prove that W_1, W_2 are independent, too. Hence, the probability that a correct decision is made, assuming that $s_1(t)$ was transmitted is :

$$\begin{array}{rcl} P_{c|s1} & = & P[W_1 > 0]\, P[W_2 > 0] = (P[W_1 > 0])^2 \\ & = & (1 - P[W_1 < 0])^2 = \left(1 - Q\left(\frac{2\mathcal{E}}{\sqrt{4\mathcal{E}N_0}}\right)\right)^2 \\ & = & \left(1 - Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)\right)^2 = \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 \end{array}$$

where $\mathcal{E}_b = \mathcal{E}/2$ is the transmitted energy per bit. Then :

$$P_{e|s1} = 1 - P_{c|s1} = 1 - \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]$$

This is the exact symbol error probability for the 4-PSK signal, which is expected since the vector space representations of the 4-biorthogonal and 4-PSK signals are identical.

Problem 5.15 :

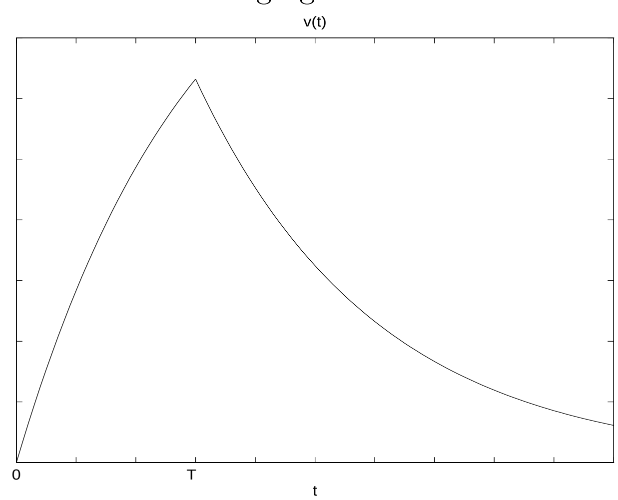
(a) The output of the matched filter can be expressed as :

$$y(t) = Re\left[v(t)e^{j2\pi f_c t}\right]$$

where $v(t)$ is the lowpass equivalent of the output :

$$v(t) = \int_0^t s_0(\tau)h(t-\tau)d\tau = \left\{ \begin{array}{ll} \int_0^t Ae^{-(t-\tau)/T}d\tau = AT\left(1-e^{-t/T}\right), & 0 \leq t \leq T \\ \int_0^T Ae^{-(t-\tau)/T}d\tau = AT(e-1)e^{-t/T}, & T \leq t \end{array} \right\}$$

(b) A sketch of $v(t)$ is given in the following figure :



(c) $y(t) = v(t)\cos 2\pi f_c t$, where $f_c >> 1/T$. Hence the maximum value of y corresponds to the maximum value of v, or $y_{\max} = y(T) = v_{\max} = v(T) = AT(1-e^{-1})$.

(d) Working with lowpass equivalent signals, the noise term at the sampling instant will be :

$$v_N(T) = \int_0^T z(\tau)h(T-\tau)d\tau$$

The mean is : $E\left[v_N(T)\right] = \int_0^T E\left[z(\tau)\right]h(T-\tau)d\tau = 0$, and the second moment :

$$\begin{array}{rcl} E\left[|v_N(T)|^2\right] & = & E\left[\int_0^T z(\tau)h(T-\tau)d\tau \int_0^T z^*(w)h(T-w)dw\right] \\ & = & 2N_0 \int_0^T h^2(T-\tau)d\tau \\ & = & N_0 T\left(1-e^{-2}\right) \end{array}$$

The variance of the real-valued noise component can be obtained using the relationship $Re[N] = \frac{1}{2}(N + N^*)$ to obtain : $\sigma_{Nr}^2 = \frac{1}{2}E\left[|v_N(T)|^2\right] = \frac{1}{2}N_0 T(1 - e^{-2})$

(e) The SNR is defined as :

$$\gamma = \frac{|v_{\max}|^2}{E\left[|v_N(T)|^2\right]} = \frac{A^2 T}{N_0}\frac{e-1}{e+1}$$

(the same result is obtained if we consider the real bandpass signal, when the energy term has the additional factor 1/2 compared to the lowpass energy term, and the noise term is $\sigma_{Nr}^2 = \frac{1}{2}E\left[|v_N(T)|^2\right]$)

(f) If we have a filter matched to $s_0(t)$, then the output of the noise-free matched filter will be :

$$v_{\max} = v(T) = \int_0^T s_o^2(t) = A^2 T$$

and the noise term will have second moment :

$$\begin{aligned} E\left[|v_N(T)|^2\right] &= E\left[\int_0^T z(\tau)s_0(T-\tau)d\tau \int_0^T z^*(w)s_0(T-w)dw\right] \\ &= 2N_0 \int_0^T s_0^2(T-\tau)d\tau \\ &= 2N_0 A^2 T \end{aligned}$$

giving an SNR of :

$$\gamma = \frac{|v_{\max}|^2}{E\left[|v_N(T)|^2\right]} = \frac{A^2 T}{2N_0}$$

Compared with the result we obtained in (e), using a sub-optimum filter, the loss in SNR is equal to : $\left(\frac{e-1}{e+1}\right)\left(\frac{1}{2}\right)^{-1} = 0.925$ or approximately 0.35 dB

Problem 5.16 :

(a) Consider the QAM constellation of Fig. P5-16. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

(b) If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

(c) The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

Problem 5.17 :

(a) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, (e.g. going in a clockwise direction : 000, 001, 011, 010, 110, 111, 101, 100). this is not the case for the 8-QAM constellation of Figure P5-16. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0,0,\ldots,0\}$ to point A, then point B and C should have the form

$$\text{B} = \{0,\ldots,0,1,0,\ldots,0\} \qquad \text{C} = \{0,\ldots,0,1,0,\ldots,0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

(b) Since each symbol conveys 3 bits of information, the resulted symbol rate is :

$$R_s = \frac{90 \times 10^6}{3} = 30 \times 10^6 \text{ symbols/sec}$$

Problem 5.18 :

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 5.19 :

(a) The PDF of the noise n is :

$$p(n) = \frac{\lambda}{2}e^{-\lambda|n|}$$

where $\lambda = \frac{\sqrt{2}}{\sigma}$ The optimal receiver uses the criterion :

$$\frac{p(r|A)}{p(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

The average probability of error is :

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{1}{2}e^{-\lambda A} = \frac{1}{2}e^{\frac{-\sqrt{2}A}{\sigma}}
\end{aligned}$$

(b) The variance of the noise is :

$$\begin{aligned}\sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\ &= \lambda\int_0^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2} = \sigma^2\end{aligned}$$

Hence, the SNR is:

$$\text{SNR} = \frac{A^2}{\sigma^2}$$

and the probability of error is given by:

$$P(e) = \frac{1}{2}e^{-\sqrt{2SNR}}$$

For $P(e) = 10^{-5}$ we obtain:

$$\ln(2\times 10^{-5}) = -\sqrt{2SNR} \Longrightarrow SNR = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then the probability of error for antipodal signalling is:

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{SNR}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{SNR} = 4.26$ and therefore SNR $= 18.1476 = 12.594$ dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 5.20 :

The constellation of Fig. P5-20(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is:

$$P_a = \frac{1}{8}\left[4\times(2A)^2 + 4\times(2\sqrt{2}A)^2\right] = 6A^2$$

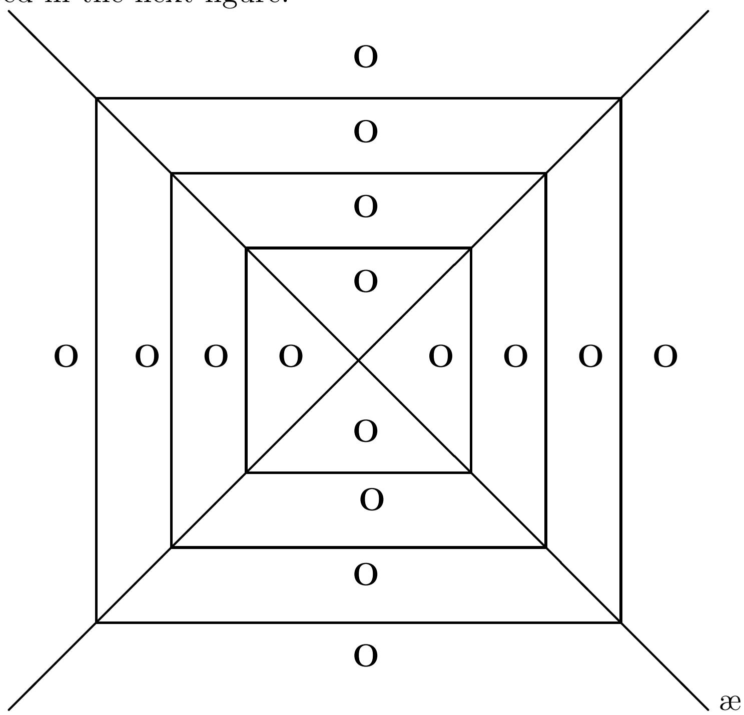
The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is:

$$P_b = \frac{1}{8}\left[4\times(\sqrt{7}A)^2 + 2\times(\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

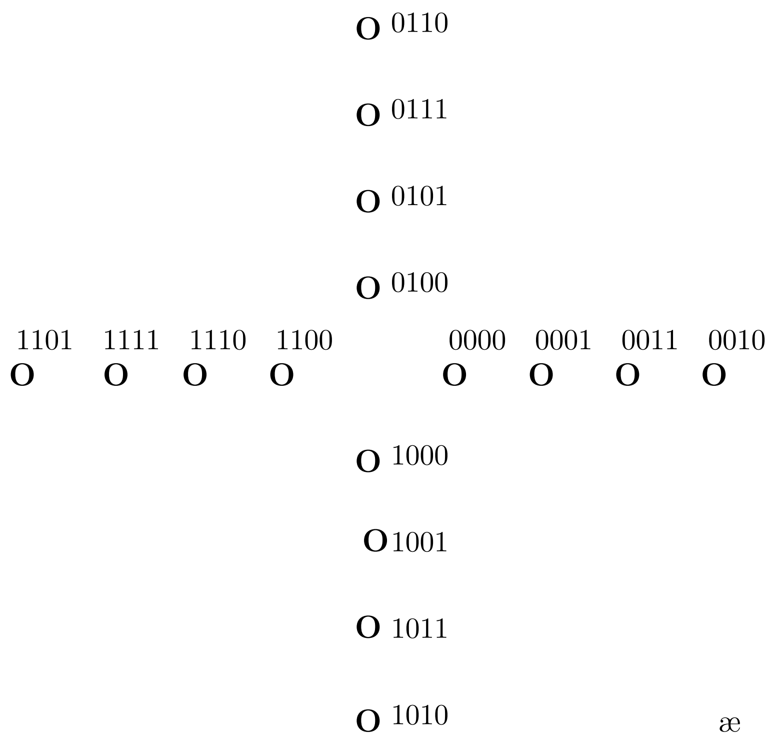
Problem 5.21 :

The optimum decision boundary of a point is determined by the perpedicular bisectors of each line segment connecting the point with its neighbors. The decision regions for this QAM constellation are depicted in the next figure:



Problem 5.22 :

One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure.



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Problem 5.23 :

The transmitted signal energy is

$$\mathcal{E}_b = \frac{A^2 T}{2}$$

where T is the bit interval and A is the signal amplitude. Since both carriers are used to transmit information over the same channel, the bit SNR, $\frac{\mathcal{E}_b}{N_0}$, is constant if A^2T is constant. Hence, the desired relation between the carrier amplitudes and the supported transmission rate $R = \frac{1}{T}$ is

$$\frac{A_c}{A_s} = \sqrt{\frac{T_s}{T_c}} = \sqrt{\frac{R_c}{R_s}}$$

With

$$\frac{R_c}{R_s} = \frac{10 \times 10^3}{100 \times 10^3} = 0.1$$

we obtain

$$\frac{A_c}{A_s} = 0.3162$$

Problem 5.24 :

(a) Since $m_2(t) = -m_3(t)$ the dimensionality of the signal space is two.

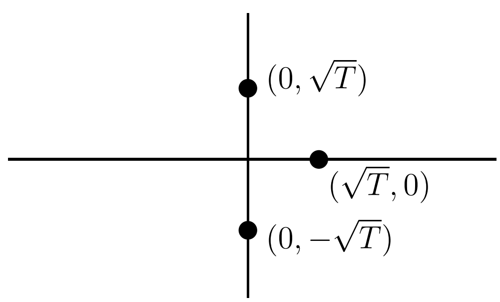
(b) As a basis of the signal space we consider the functions:

$$f_1(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \qquad f_2(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t \le \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} < t \le T \\ 0 & \text{otherwise} \end{cases}$$

The vector representation of the signals is:

$$\begin{aligned} \mathbf{m}_1 &= [\sqrt{T}, \ 0] \\ \mathbf{m}_2 &= [0, \ \sqrt{T}] \\ \mathbf{m}_3 &= [0, \ -\sqrt{T}] \end{aligned}$$

(c) The signal constellation is depicted in the next figure :



(d) The three possible outputs of the matched filters, corresponding to the three possible transmitted signals are $(r_1, r_2) = (\sqrt{T} + n_1, n_2)$, $(n_1, \sqrt{T} + n_2)$ and $(n_1, -\sqrt{T} + n_2)$, where n_1, n_2 are zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. If all the signals are equiprobable the optimum decision rule selects the signal that maximizes the metric (see 5-1-44):

$$C(\mathbf{r}, \mathbf{m}_i) = 2\mathbf{r} \cdot \mathbf{m}_i - |\mathbf{m}_i|^2$$

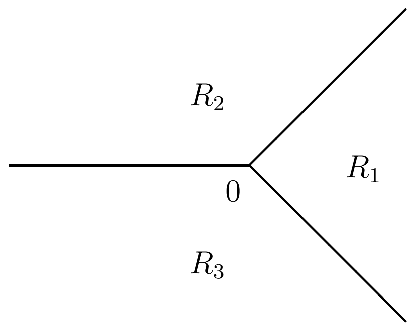
or since $|\mathbf{m}_i|^2$ is the same for all i,

$$C'(\mathbf{r}, \mathbf{m}_i) = \mathbf{r} \cdot \mathbf{m}_i$$

Thus the optimal decision region R_1 for $\mathbf{m}_1$ is the set of points (r_1, r_2), such that $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_2$ and $(r_1, r_2) \cdot \mathbf{m}_1 > (r_1, r_2) \cdot \mathbf{m}_3$. Since $(r_1, r_2) \cdot \mathbf{m}_1 = \sqrt{T} r_1$, $(r_1, r_2) \cdot \mathbf{m}_2 = \sqrt{T} r_2$ and $(r_1, r_2) \cdot \mathbf{m}_3 = -\sqrt{T} r_2$, the previous conditions are written as

$$r_1 > r_2 \quad \text{and} \quad r_1 > -r_2$$

Similarly we find that R_2 is the set of points (r_1, r_2) that satisfy $r_2 > 0$, $r_2 > r_1$ and R_3 is the region such that $r_2 < 0$ and $r_2 < -r_1$. The regions R_1, R_2 and R_3 are shown in the next figure.



(e) If the signals are equiprobable then:

$$P(e|\mathbf{m}_1) = P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_2|^2 | \mathbf{m}_1) + P(|\mathbf{r} - \mathbf{m}_1|^2 > |\mathbf{r} - \mathbf{m}_3|^2 | \mathbf{m}_1)$$

When $\mathbf{m}_1$ is transmitted then $\mathbf{r} = [\sqrt{T} + n_1, n_2]$ and therefore, $P(e|\mathbf{m}_1)$ is written as:

$$P(e|\mathbf{m}_1) = P(n_2 - n_1 > \sqrt{T}) + P(n_1 + n_2 < -\sqrt{T})$$

Since, n_1, n_2 are zero-mean statistically independent Gaussian random variables, each with variance $\frac{N_0}{2}$, the random variables $x = n_1 - n_2$ and $y = n_1 + n_2$ are zero-mean Gaussian with variance N_0. Hence:

$$\begin{aligned} P(e|\mathbf{m}_1) &= \frac{1}{\sqrt{2\pi N_0}} \int_{\sqrt{T}}^{\infty} e^{-\frac{x^2}{2N_0}} dx + \frac{1}{\sqrt{2\pi N_0}} \int_{-\infty}^{-\sqrt{T}} e^{-\frac{y^2}{2N_0}} dy \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{T}{N_0}}\right] = 2Q\left[\sqrt{\frac{T}{N_0}}\right] \end{aligned}$$

When $\mathbf{m}_2$ is transmitted then $\mathbf{r} = [n_1, n_2 + \sqrt{T}]$ and therefore:

$$\begin{aligned} P(e|\mathbf{m}_2) &= P(n_1 - n_2 > \sqrt{T}) + P(n_2 < -\sqrt{T}) \\ &= Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right] \end{aligned}$$

Similarly from the symmetry of the problem, we obtain:

$$P(e|\mathbf{m}_2) = P(e|\mathbf{m}_3) = Q\left[\sqrt{\frac{T}{N_0}}\right] + Q\left[\sqrt{\frac{2T}{N_0}}\right]$$

Since $Q[\cdot]$ is momononically decreasing, we obtain:

$$Q\left[\sqrt{\frac{2T}{N_0}}\right] < Q\left[\sqrt{\frac{T}{N_0}}\right]$$

and therefore, the probability of error $P(e|\mathbf{m}_1)$ is larger than $P(e|\mathbf{m}_2)$ and $P(e|\mathbf{m}_3)$. Hence, the message $\mathbf{m}_1$ is more vulnerable to errors. The reason for that is that it has both threshold lines close to it, while the other two signals have one of the their threshold lines further away.

Problem 5.25 :

(a) If the power spectral density of the additive noise is $\mathcal{S}_n(f)$, then the PSD of the noise at the output of the prewhitening filter is

$$\mathcal{S}_\nu(f) = \mathcal{S}_n(f)|H_p(f)|^2$$

In order for $\mathcal{S}_\nu(f)$ to be flat (white noise), $H_p(f)$ should be such that

$$H_p(f) = \frac{1}{\sqrt{\mathcal{S}_n(f)}}$$

(b) Let $h_p(t)$ be the impulse response of the prewhitening filter $H_p(f)$. That is, $h_p(t) = \mathcal{F}^{-1}[H_p(f)]$. Then, the input to the matched filter is the signal $\tilde{s}(t) = s(t) \star h_p(t)$. The frequency response of the filter matched to $\tilde{s}(t)$ is

$$\tilde{S}_m(f) = \tilde{S}^*(f)e^{-j2\pi f t_0} == S^*(f)H_p^*(f)e^{-j2\pi f t_0}$$

where t_0 is some nominal time-delay at which we sample the filter output.

(c) The frequency response of the overall system, prewhitenig filter followed by the matched filter, is

$$G(f) = \tilde{S}_m(f)H_p(f) = S^*(f)|H_p(f)|^2 e^{-j2\pi f t_0} = \frac{S^*(f)}{\mathcal{S}_n(f)}e^{-j2\pi f t_0}$$

(d) The variance of the noise at the output of the generalized matched filter is

$$\sigma^2 = \int_{-\infty}^{\infty} \mathcal{S}_n(f)|G(f)|^2 df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)}df$$

At the sampling instant $t = t_0 = T$, the signal component at the output of the matched filter is

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty} Y(f)e^{j2\pi f T}df = \int_{-\infty}^{\infty} s(\tau)g(T-\tau)d\tau \\ &= \int_{-\infty}^{\infty} S(f)\frac{S^*(f)}{\mathcal{S}_n(f)}df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)}df \end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{\mathcal{S}_n(f)}df$$

Problem 5.26 :

(a) The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 5 \times 10^{-6} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 9.7682$$

(b) If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{4}\right) Q\left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}}\right]\right)^2$$

Thus,

$$Q\left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}}\right] = \frac{1}{3} \times 10^{-5} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 25.3688$$

(c) If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{16}\right) Q\left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

(d) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

Problem 5.27 :

Using the Pythagorean theorem for the four-phase constellation, we find:

$$r_1^2 + r_1^2 = d^2 \Longrightarrow r_1 = \frac{d}{\sqrt{2}}$$

The radius of the 8-PSK constellation is found using the cosine rule. Thus:

$$d^2 = r_2^2 + r_2^2 - 2r_2^2\cos(45^o) \Longrightarrow r_2 = \frac{d}{\sqrt{2-\sqrt{2}}}$$

The average transmitted power of the 4-PSK and the 8-PSK constellation is given by:

$$P_{4,av} = \frac{d^2}{2}, \qquad P_{8,av} = \frac{d^2}{2-\sqrt{2}}$$

Thus, the additional transmitted power needed by the 8-PSK signal is:

$$P = 10\log_{10}\frac{2d^2}{(2-\sqrt{2})d^2} = 5.3329 \text{ dB}$$

We obtain the same results if we use the probability of error given by (see 5-2-61) :

$$P_M = 2Q\left[\sqrt{2\gamma_s}\sin\frac{\pi}{M}\right]$$

where γ_s is the SNR per symbol. In this case, equal error probability for the two signaling schemes, implies that

$$\gamma_{4,s}\sin^2\frac{\pi}{4} = \gamma_{8,s}\sin^2\frac{\pi}{8} \Longrightarrow 10\log_{10}\frac{\gamma_{8,s}}{\gamma_{4,s}} = 20\log_{10}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}} = 5.3329 \text{ dB}$$

Since we consider that error ocuur only between adjacent points, the above result is equal to the additional transmitted power we need for the 8-PSK scheme to achieve the same distance d between adjacent points.

Problem 5.28 :

For 4-phase PSK ($M = 4$) we have the following realtionship between the symbol rate $1/T$, the required bandwith W and the bit rate $R = k \cdot 1/T = \frac{log_2 M}{T}$ (see 5-2-84):

$$W = \frac{R}{log_2 M} \rightarrow R = W log_2 M = 2W = 200 \text{ kbits/sec}$$

For binary FSK ($M = 2$) the required frequency separation is $1/2T$ (assuming coherent receiver) and (see 5-2-86):

$$W = \frac{M}{log_2 M} R \rightarrow R = \frac{2W log_2 M}{M} = W = 100 \text{ kbits/sec}$$

Finally, for 4-frequency non-coherent FSK, the required frequency separation is $1/T$, so the symbol rate is half that of binary coherent FSK, but since we have two bits/symbol, the bit ate is tha same as in binary FSK :

$$R = W = 100 \text{ kbits/sec}$$

Problem 5.29 :

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant n is given by

$$\theta(n; \mathbf{a}) = \frac{\pi}{2} \sum_{k=0}^{k} a_k + \theta_0$$

where θ_0 is the initial phase and a_k is ± 1 depending on the input bit at the time instant k. The following table shows $\theta(n; \mathbf{a})$ for two different values of θ_0 $(0, \pi)$, and the four input pairs of data: $\{00, 01, 10, 11\}$.

θ_0	b_0	b_1	a_0	a_1	$\theta(n; \mathbf{a})$
0	0	0	-1	-1	$-\pi$
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

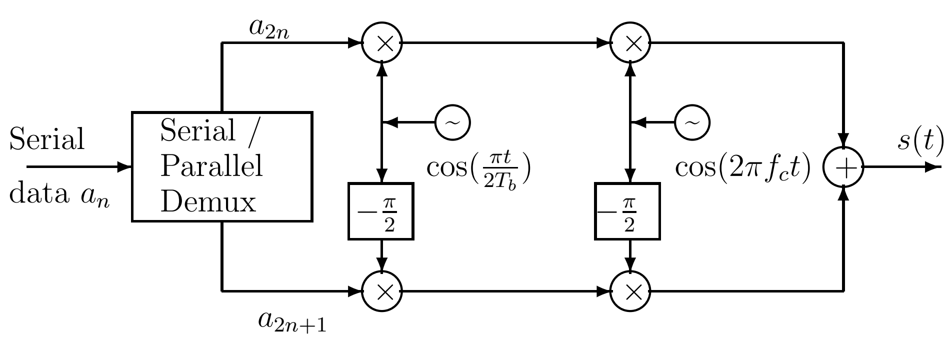
Problem 5.30 :

(a) The envelope of the signal is

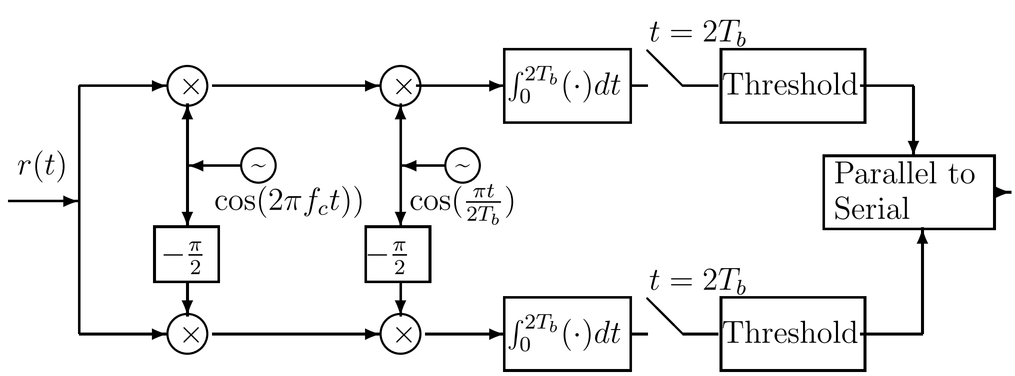
$$\begin{aligned} |s(t)| &= \sqrt{|s_c(t)|^2 + |s_s(t)|^2} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}\cos^2\left(\frac{\pi t}{2T_b}\right) + \frac{2\mathcal{E}_b}{T_b}\sin^2\left(\frac{\pi t}{2T_b}\right)} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}} \end{aligned}$$

Thus, the signal has constant amplitude.

(b) The signal $s(t)$ is equivalent to an MSK signal. A block diagram of the modulator for synthesizing the signal is given in the next figure.



(c) A sketch of the demodulator is shown in the next figure.



Problem 5.31 :

$$h = \frac{1}{2}, \quad L = 2$$

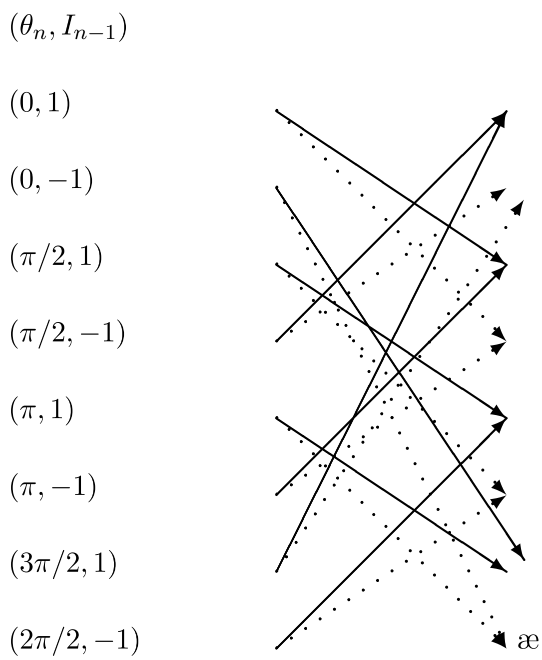
Based on (5-3-7), we obtain the 4 phase states :

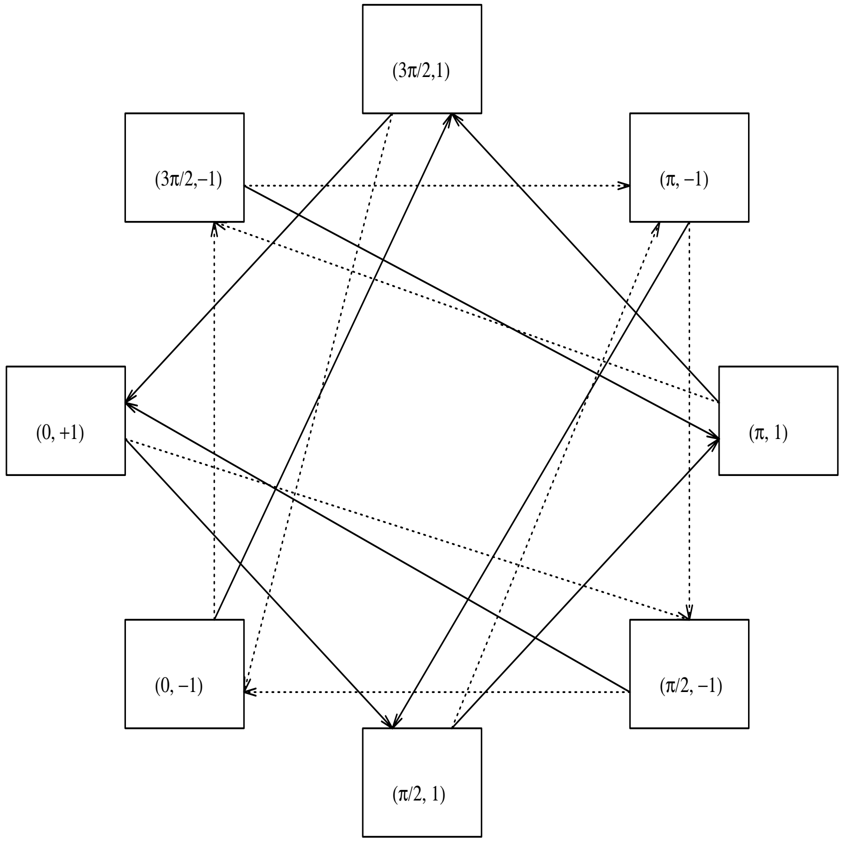
$$\Theta_s = \{0, \pi/2, \pi, 3\pi/2\}$$

The states in the trellis are the combination of the phase state and the correlative state, which take the values $I_{n-1} = \{\pm 1\}$. The transition from state to state are determined by

$$\theta_{n+1} = \theta_n + \frac{\pi}{2} I_{n-1}$$

and the resulting state trellis and state diagram are given in the following figures, where a solid line corresponds to $I_n = 1$, while a dotted line corresponds to $I_n = -1$.





The treatment in Probl. 4.27 involved the terminal phase states only, which were determined to be $\{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$. We can easily verify that each two of the combined states, which were obtained in this problem, give one terminal phase state. For example $(\theta_n, I_{n-1}) = (3\pi/2, -1)$ and $(\theta_n, I_{n-1}) = (\pi, +1)$, give the same terminal phase state at $t = (n+1)T$:

$$\phi\left((n+1)T; \mathbf{I}\right) = \theta_n + \theta(t; \mathbf{I}) = \theta_n + 2\pi h I_{n-1} q(2T) + 2\pi h I_n q(T) \Rightarrow$$
$$\phi\left((n+1)T; \mathbf{I}\right) = \theta_n + \tfrac{\pi}{2} I_{n-1} + \tfrac{\pi}{4} I_n = \tfrac{3\pi}{2} + \tfrac{\pi}{4} I_n = 5\pi/4 \text{ or } 7\pi/4$$

Problem 5.32 :

(a)

(i) There are no correlative states in this system, since it is a full response CPM. Based on (5-3-6), we obtain the phase states :

$$\Theta_s = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

(ii) Based on (5-3-7), we obtain the phase states :

$$\Theta_s = \left\{0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{9\pi}{4} \equiv \frac{\pi}{4}, \pi, \frac{15\pi}{4} \equiv \frac{7\pi}{4}, \frac{18\pi}{4} \equiv \frac{\pi}{2}, \frac{21\pi}{4} \equiv \frac{5\pi}{4}\right\}$$

(b)
(i) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\left\{, I_{n-1/n-2}\right\}$ take the values ± 1. Hence there are $3 \times 2 \times 2 = 12$ combined states in all.
(ii) The combined states are $S_n = (\theta_n, I_{n-1}, I_{n-2})$, where $\left\{, I_{n-1/n-2}\right\}$ take the values ± 1. Hence there are $8 \times 2 \times 2 = 32$ combined states in all.

Problem 5.33 :

A biorthogonal signal set with $M = 8$ signal points has vector space dimensionallity 4. Hence, the detector first checks which one of the four correlation metrics is the largest in absolute value, and then decides about the two possible symbols associated with this correlation metric,based on the sign of this metric. Hence, the error probability is the probability of the union of the event E_1 that another correlation metric is greater in absolute value and the event E_2 that the signal correlation metric has the wrong sign. A union bound on the symbol error probability can be given by :

$$P_M \leq P(E_1) + P(E_2)$$

But $P(E_2)$ is simply the probability of error for an antipodal signal set : $P(E_2) = Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)$ and the probability of the event E_1 can be union bounded by :

$$P(E_1) \leq 3\left[P(|C_2| > |C_1|)\right] = 3\left[2P(C_2 > C_1)\right] = 6P\left(C_2 > C_1\right) = 6Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$$

where C_i is the correlation metric corresponding to the i-th vector space dimension; the probability that a correlation metric is greater that the correct one is given by the error probability for orthogonal signals $Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$(since these correlation metrics correspond to orthogonal signals). Hence :

$$P_M \leq 6Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) + Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)$$

(sum of the probabilities to chose one of the 6 orthogonal, to the correct one, signal points and the probability to chose the signal point which is antipodal to the correct one).

Problem 5.34 :

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable,

$$P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|\mathbf{s}_i)$$

All the $P(C|\mathbf{s}_i)$, $i = 1, \ldots, M$ are identical because of the symmetry of the constellation. By translating the vector $\mathbf{s}_i$ to the origin we can find the probability of a correct decison, given that $\mathbf{s}_i$ was transmitted, as :

$$P(C|\mathbf{s}_i) = \int_{-\frac{d}{2}}^{\infty} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\infty} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\infty} f(n_N)dn_N$$

where the number of the integrals on the right side of the equation is N, d is the minimum distance between the points and :

$$f(n_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_i^2}{2\sigma^2}} = \frac{1}{\sqrt{\pi N_0}} e^{-\frac{n_i^2}{N_0}}$$

Hence :

$$\begin{aligned} P(C|\mathbf{s}_i) &= \left(\int_{-\frac{d}{2}}^{\infty} f(n)dn \right)^N = \left(1 - \int_{-\infty}^{-\frac{d}{2}} f(n)dn \right)^N \\ &= \left(1 - Q\left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \end{aligned}$$

and therefore, the probability of error is given by :

$$\begin{aligned} P(e) &= 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M} \left(1 - Q\left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \\ &= 1 - \left(1 - Q\left[\frac{d}{\sqrt{2N_0}} \right] \right)^N \end{aligned}$$

Note that since :

$$\mathcal{E}_s = \sum_{i=1}^{N} s_{m,i}^2 = \sum_{i=1}^{N} (\frac{d}{2})^2 = N\frac{d^2}{4}$$

the probability of error can be written as :

$$P(e) = 1 - \left(1 - Q\left[\sqrt{\frac{2\mathcal{E}_s}{NN_0}} \right] \right)^N$$

Problem 5.35 :

Consider first the signal :

$$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is :

$$\begin{aligned} g(t) &= y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^{n} c_{n-i+1} \delta((i-1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1} \delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$. Since,

$$s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) \star \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is :

$$S(f) = P(f) \sum_{k=1}^{n} c_k e^{-j2\pi f k T_c}$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is :

$$\begin{aligned} H(f) &= S^*(f) e^{-j2\pi f T} = S^*(f) e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{k=1}^{n} c_k e^{j2\pi f k T_c} e^{-j2\pi f n T_c} \\ &= P^*(f) \sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f (i-1) T_c} \\ &= P^*(f) \mathcal{F}[g(t)] \end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal $y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is (see 5-1-27) :

$$\begin{aligned} \int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c) dt \\ &= T_c \sum_{k=1}^{n} c_k^2 \end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Problem 5.36 :

The bandwidth required for transmission of an M-ary PAM signal is

$$W = \frac{R}{2 \log_2 M} \text{ Hz}$$

Since,

$$R = 8 \times 10^3 \frac{\text{samples}}{\text{sec}} \times 8 \frac{\text{bits}}{\text{sample}} = 64 \times 10^3 \frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16 \text{ KHz} & M = 4 \\ 10.667 \text{ KHz} & M = 8 \\ 8 \text{ KHz} & M = 16 \end{cases}$$

Problem 5.37 :

(a) The inner product of $s_i(t)$ and $s_j(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} s_i(t)s_j(t)dt &= \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik}p(t-kT_c) \sum_{l=1}^{n} c_{jl}p(t-lT_c)dt \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl} \int_{-\infty}^{\infty} p(t-kT_c)p(t-lT_c)dt \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl}\mathcal{E}_p\delta_{kl} \\ &= \mathcal{E}_p \sum_{k=1}^{n} c_{ik}c_{jk} \end{aligned}$$

The quantity $\sum_{k=1}^{n} c_{ik}c_{jk}$ is the inner product of the row vectors $\underline{C}_i$ and $\underline{C}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2\delta_{ij} = n\mathcal{E}_p\delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

(b) Using the results of Problem 5.35, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik}p(t-kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{C}_i = [c_{i1}, \ldots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the n matched filters can be realized by a filter matched to $p(t)$ followed by n discrete-time filters matched to the vectors $\underline{C}_i$, $i = 1, \ldots, n$.

Problem 5.38 :

(a) The optimal ML detector (see 5-1-41) selects the sequence $\underline{C}_i$ that minimizes the quantity:

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n-w$ received elements of $\mathbf{r}$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \underset{\underline{C}_1}{\overset{\underline{C}_2}{\gtrless}} 0$$

or equivalently

$$\sum_{k=w+1}^{n} r_k \underset{\underline{C}_2}{\overset{\underline{C}_1}{\gtrless}} 0$$

(b) Since $r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^{n} n_k < 0\right) \\ &= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b}\right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp(-\frac{x^2}{2\pi(n-w)\sigma^2}) dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

(c) The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 5.39 :

$r_l(t) = s_{l1}(t)e^{j\phi} + z(t)$. Hence, the output of a correlator-type receiver will be :

$$\begin{aligned} r_1 &= \int_0^T r_l(t)s_{l1}^*(t)dt = \int_0^T \left(s_{l1}(t)e^{j\phi} + z(t)\right) s_{l1}^*(t)dt \\ &= e^{j\phi}\int_0^T s_{l1}(t)s_{l1}^*(t)dt + \int_0^T z(t)s_{l1}^*(t)dt \\ &= e^{j\phi}2\mathcal{E} + n_{1c} + jn_{1s} = 2\mathcal{E}\cos\phi + n_{1c} + j\left(2\mathcal{E}\sin\phi + n_{1s}\right) \end{aligned}$$

where $n_{1c} = Re\left[\int_0^T z(t)s_{l1}^*(t)dt\right],\ n_{1s} = Im\left[\int_0^T z(t)s_{l1}^*(t)dt\right]$. Similarly for the second correlator output:

$$\begin{aligned} r_2 &= \int_0^T r_l(t)s_{l2}^*(t)dt = \int_0^T \left(s_{l1}(t)e^{j\phi} + z(t)\right) s_{l2}^*(t)dt \\ &= e^{j\phi}\int_0^T s_{l1}(t)s_{l2}^*(t)dt + \int_0^T z(t)s_{l2}^*(t)dt \\ &= e^{j\phi}2\mathcal{E}\rho^* + n_{2c} + jn_{2s} = e^{j\phi}2\mathcal{E}\left|\rho\right|e^{-ja_0} + n_{2c} + jn_{2s} \\ &= 2\mathcal{E}\left|\rho\right|\cos\left(\phi - a_0\right) + n_{2c} + j\left(2\mathcal{E}\left|\rho\right|\sin\left(\phi - a_0\right) + n_{2s}\right) \end{aligned}$$

where $n_{2c} = Re\left[\int_0^T z(t)s_{l2}^*(t)dt\right],\ n_{2s} = Im\left[\int_0^T z(t)s_{l2}^*(t)dt\right]$.

Problem 5.40 :

$n_{ic} = Re\left[\int_0^T z(t)s_{li}^*(t)dt\right],\ n_{is} = Im\left[\int_0^T z(t)s_{li}^*(t)dt\right], \quad i = 1, 2$. The variances of the noise terms are:

$$\begin{aligned} E\left[n_{1c}n_{1c}\right] &= E\left[Re\left[\int_0^T z(t)s_{l1}^*(t)dt\right] Re\left[\int_0^T z(t)s_{l1}^*(t)dt\right]\right] \\ &= \frac{1}{4}E\left[\left(\int_0^T z(t)s_{l1}^*(t)dt + \int_0^T z^*(t)s_{l1}(t)dt\right)^2\right] \\ &= \frac{1}{4}2\int_0^T\int_0^T E\left[z(a)z^*(t)\right]s_{l1}(t)s_{l1}^*(a)dtda \\ &= \frac{1}{4}2\cdot 2N_0\int_0^T s_{l1}(t)s_{l1}^*(t)dt = 2N_0\mathcal{E} \end{aligned}$$

where we have used the identity $Re\left[z\right] = \frac{1}{2}\left(z+z^*\right)$, and the fact from Problem 5.7 (or 4.3) that $E\left[z(t)z(t+\tau)\right]=0,\ E\left[z^*(t)z^*(t+\tau)\right]=0$. Similarly :

$$E\left[n_{2c}n_{2c}\right] = E\left[n_{1s}n_{1s}\right] = E\left[n_{2s}n_{2s}\right] = 2N_0\mathcal{E}$$

where for the quadrature noise term components we use the identity : $Im\left[z\right] = \frac{1}{2j}\left(z-z^*\right)$.
The covariance between the in-phase terms for the two correlators is :

$$\begin{aligned} E\left[n_{1c}n_{2c}\right] &= \frac{1}{4}E\left[\left(\int_0^T z(t)s_{l1}^*(t)dt + \int_0^T z^*(t)s_{l1}(t)dt\right)\left(\int_0^T z(t)s_{l2}^*(t)dt + \int_0^T z^*(t)s_{l2}(t)dt\right)\right] \\ &= 0 \end{aligned}$$

because the 4 crossterms that are obtained from the above expressions contain one of :

$$E\left[z(t)z(t+\tau)\right] = E\left[z^*(t)z^*(t+\tau)\right] = \int_0^T s_{l2}(t)s_{l1}^*(t)dt = \int_0^T s_{l1}(t)s_{l2}^*(t)dt = 0$$

Similarly : $E\left[n_{1c}n_{2s}\right] = E\left[n_{1s}n_{2s}\right] = E\left[n_{1s}n_{2c}\right] = 0$.
Finally, the covariance between the in-phase and the quadrature component of the same correlator output is :

$$\begin{aligned} E\left[n_{1c}n_{1s}\right] &= \frac{1}{4j}E\left[\left(\int_0^T z(t)s_{l1}^*(t)dt + \int_0^T z^*(t)s_{l1}(t)dt\right)\left(\int_0^T z(t)s_{l1}^*(t)dt - \int_0^T z^*(t)s_{l1}(t)dt\right)\right] \\ &= \frac{1}{4j}\left(\int_0^T\int_0^T E\left[z(a)z(t)\right]s_{l1}^*(a)s_{l1}^*(t)dtda + \int_0^T\int_0^T E\left[z^*(a)z^*(t)\right]s_{l1}(a)s_{l1}^*dtda\right) \\ &\quad +\frac{1}{4j}\left(\int_0^T\int_0^T E\left[z^*(a)z(t)\right]s_{l1}(a)s_{l1}^*(t)dtda - \int_0^T\int_0^T E\left[z^*(a)z(t)\right]s_{l1}(a)s_{l1}^*(t)dtda\right) \\ &= \frac{1}{4j}\left(\int_0^T\int_0^T E\left[z(a)z(t)\right]s_{l1}^*(a)s_{l1}^*(t)dtda + \int_0^T\int_0^T E\left[z^*(a)z^*(t)\right]s_{l1}(a)s_{l1}^*dtda\right) \\ &= 0 \end{aligned}$$

Similarly : $E\left[n_{2c}n_{2s}\right] = 0$.
The joint pdf is simply the product of the marginal pdf's, since these noise terms are Gaussian and uncorrelated, and thus they are also statistically independent :

$$p(n_{1c},n_{2c},n_{1s},n_{2s}) = p(n_{1c})p(n_{2c})p(n_{1s})p(n_{2s})$$

where, e.g : $p(n_{1c}) = \frac{1}{\sqrt{4\pi N_o \mathcal{E}}} \exp(-n_{1c}^2/4N_0\mathcal{E})$.

Problem 5.41 :

The first matched filter output is :

$$r_1 = \int_0^T r_l(\tau)h_1(T-\tau)d\tau = \int_0^T r_l(\tau)s_{l1}^*(T-(T-\tau))d\tau = \int_0^T r_l(\tau)s_{l1}^*(\tau)d\tau$$

Similarly :

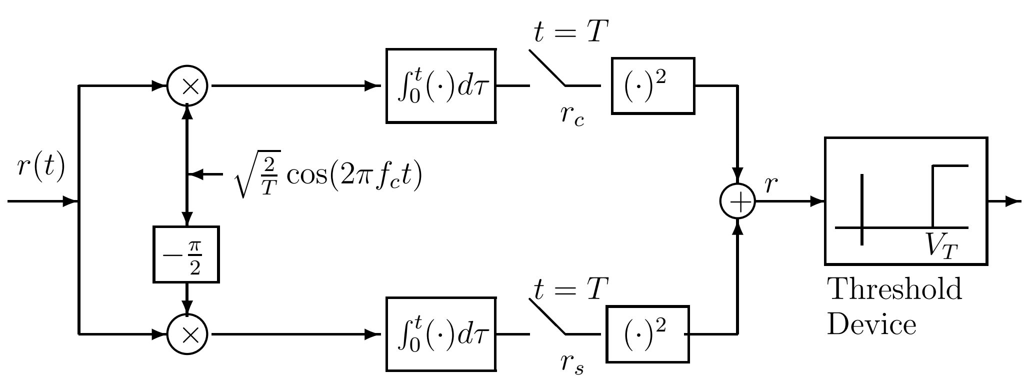
$$r_2 = \int_0^T r_l(\tau)h_2(T-\tau)d\tau = \int_0^T r_l(\tau)s_{l2}^*(T-(T-\tau))d\tau = \int_0^T r_l(\tau)s_{l2}^*(\tau)d\tau$$

which are the same as those of the correlation-type receiver of Problem 5.39. From this point, following the exact same steps as in Problem 5.39, we get :

$$\begin{array}{rcl} r_1 &=& 2\mathcal{E}\cos\phi + n_{1c} + j\,(2\mathcal{E}\sin\phi + n_{1s}) \\ r_2 &=& 2\mathcal{E}\,|\rho|\cos(\phi - a_0) + n_{2c} + j\,(2\mathcal{E}\,|\rho|\sin(\phi - a_0) + n_{2s}) \end{array}$$

Problem 5.42 :

(a) The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



(b) If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by :

$$p(r|s_0(t)) = \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0}e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is :

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned} r_c &= \int_0^T r(t)\sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\ &= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t)\sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\ &= \frac{2\sqrt{\mathcal{E}_b}}{T_b} \int_0^T \frac{1}{2} \left(\cos(2\pi 2 f_c t + \phi) + \cos(\phi)\right) dt + n_c \\ &= \sqrt{\mathcal{E}_b} \cos(\phi) + n_c \end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have :

$$r_s = \sqrt{\mathcal{E}_b} \sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ follows the Rician distibution :

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+\mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

(c) For equiprobable signals the probability of error is given by:

$$P(\text{error}) = \frac{1}{2}\int_{-\infty}^{V_T} p(r|s_1(t)) dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t)) dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned} P(\text{error}) &= \frac{1}{2}\int_0^{V_T} p(r|s_1(t)) dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t)) dr \\ &= \frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2}\int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr \end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to: $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation :

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case :

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore :

$$\begin{aligned}
\frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\
&= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right]
\end{aligned}$$

Finally :

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\
&\leq \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} e^{-\frac{\mathcal{E}_b}{4N_0}}
\end{aligned}$$

Problem 5.43 :

(a) $D = Re\left(V_m V_{m-1}^*\right)$ where $V_m = X_m + jY_m$. Then :

$$\begin{aligned}
D &= Re\left((X_m + jY_m)(X_{m-1} - jY_{m-1})\right) \\
&= X_m X_{m-1} + Y_m Y_{m-1} \\
&= \left(\frac{X_m + X_{m-1}}{2}\right)^2 - \left(\frac{X_m - X_{m-1}}{2}\right)^2 + \left(\frac{Y_m + Y_{m-1}}{2}\right)^2 - \left(\frac{Y_m - Y_{m-1}}{2}\right)^2
\end{aligned}$$

(b) $V_k = X_k + jY_k = 2aE\cos(\theta - \phi) + j2aE\sin(\theta - \phi) + N_{k,real} + N_{k,imag}$. Hence :

$$\begin{array}{ll}
U_1 = \frac{X_m + X_{m-1}}{2}, & E(U_1) = 2aE\cos(\theta - \phi) \\
U_2 = \frac{Y_m + Y_{m-1}}{2}, & E(U_2) = 2aE\sin(\theta - \phi) \\
U_3 = \frac{X_m - X_{m-1}}{2}, & E(U_3) = 0 \\
U_4 = \frac{Y_m - Y_{m-1}}{2}, & E(U_4) = 0
\end{array}$$

The variance of U_1 is : $E\left[U_1 - E(U_1)\right]^2 = E\left[\frac{1}{2}\left(N_{m,real} + N_{m-1,real}\right)\right]^2 = E\left[N_{m,real}\right]^2 = 2\mathcal{E}N_0$, and similarly : $E\left[U_i - E(U_i)\right]^2 = 2EN_0,\ i = 2,3,4$. The covariances are (e.g for U_1, U_2): $cov(U_1, U_2) = E\left[\left(U_1 - E(U_1)\right)\left(U_2 - E(U_2)\right)\right] = E\left[\frac{1}{4}\left(N_{m,r} + N_{m-1,r}\right)\left(N_{m,i} + N_{m-1,i}\right)\right] = 0$, since the noise components are uncorrelated and have zero mean. Similarly for any i, j : $cov(U_i, U_j) = 0$. The condition $cov(U_i, U_j) = 0$, implies that these random variables $\{U_i\}$ are uncorrelated, and since they are Gaussian, they are also statistically independent.
Since U_3 and U_4 are zero-mean Gaussian, the random variable $R_2 = \sqrt{U_3^2 + U_4^2}$ has a Rayleigh distribution; on the other hand $R_1 = \sqrt{U_1^2 + U_2^2}$ has a Rice distribution.

(c) $W_1 = U_1^2 + U_2^2$, with U_1, U_2 being statistically independent Gaussian variables with means $2aE\cos(\theta - \phi), 2aE(\sin\theta - \phi)$ and identical variances $\sigma^2 = 2EN_0$. Then, W_1 follows a non-central chi-square distribution with pdf given by (2-1-118):

$$p(w_1) = \frac{1}{4EN_0} e^{-\left(4a^2E^2 + w_1\right)/4EN_0} I_0\left(\frac{a}{N_0}\sqrt{w_1}\right),\ w_1 \geq 0$$

Also, $W_2 = U_3^2 + U_4^2$, with U_3, U_4 being zero-mean Gaussian with the same variance. Hence, W_1 follows a central chi-square distribution, with pfd given by (2-1-110) :

$$p(w_2) = \frac{1}{4EN_0} e^{-w_2/4EN_0},\ w_2 \geq 0$$

(d)

$$\begin{aligned}
P_b &= P(D < 0) = P(W_1 - W_2 < 0) \\
&= \int_0^\infty P(w_2 > w_1 | w_1) p(w_1) dw_1 \\
&= \int_0^\infty \left(\int_{w_1}^\infty p(w_2) dw_2\right) p(w_1) dw_1 \\
&= \int_0^\infty e^{-w_1/4EN_0} p(w_1) dw_1 \\
&= \psi(jv)|_{v=j/4EN_0} \\
&= \frac{1}{(1-2jv\sigma^2)} \exp\left(\frac{jv4a^2E^2}{1-2jv\sigma^2}\right)|_{v=j/4EN_0} \\
&= \frac{1}{2} e^{-a^2E/N_0}
\end{aligned}$$

where we have used the characteristic function of the non-central chi-square distribution given by (2-1-117) in the book.

Problem 5.44 :

$v(t) = \sum_k [I_k u(t-2kT_b) + jJ_k u(t-2kT_b-T_b)]$ where $u(t) = \left\{ \begin{array}{ll} \sin\frac{\pi t}{2T_b}, & 0 \leq t \leq T_b \\ 0, & \text{o.w.} \end{array} \right\}$. Note that

$$u(t-T_b) = \sin\frac{\pi(t-T_b)}{2T_b} = -\cos\frac{\pi t}{2T_b}, \quad T_b \leq t \leq 3T_b$$

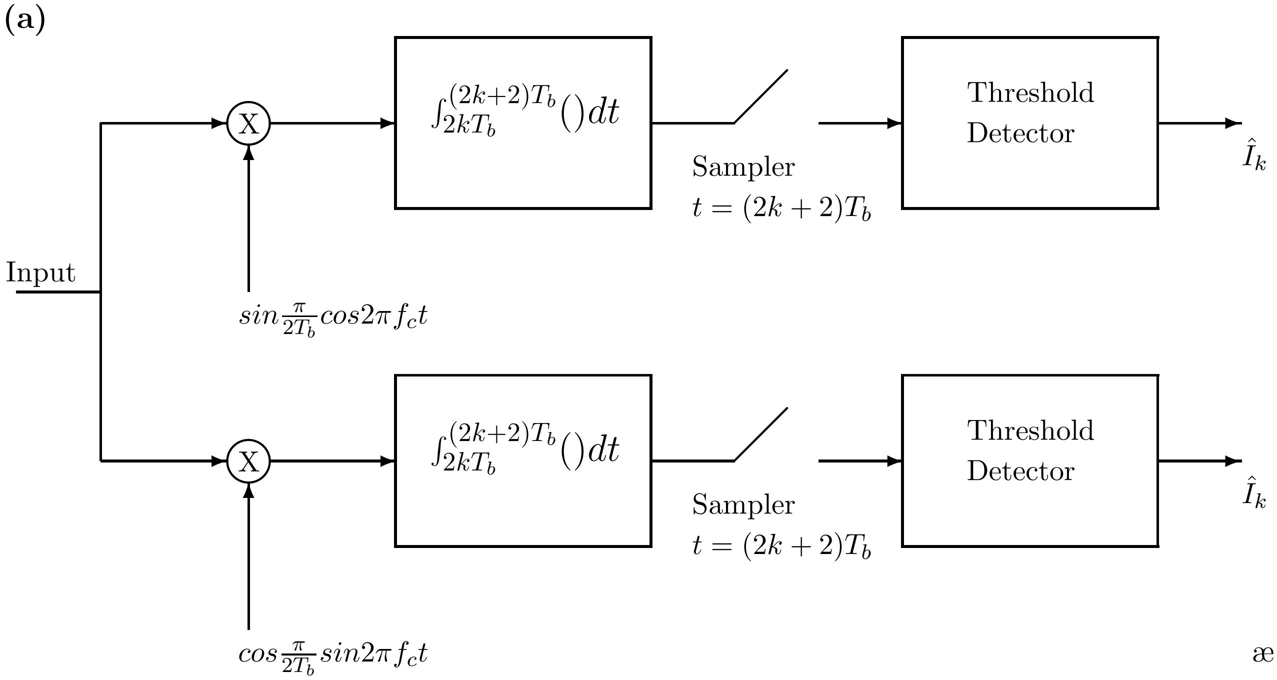
Hence, $v(t)$ may be expressed as :

$$v(t) = \sum_k \left[I_k \sin\frac{\pi(t-2kT_b)}{2T_b} - jJ_k \cos\frac{\pi(t-2kT_b)}{2T_b} \right]$$

The transmitted signal is :

$$Re\left[v(t)e^{j2\pi f_c t}\right] = \sum_k \left[I_k \sin\frac{\pi(t-2kT_b)}{2T_b}\cos 2\pi f_c t + J_k \cos\frac{\pi(t-2kT_b)}{2T_b}\sin 2\pi f_c t \right]$$

(a)



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(b) The offset QPSK signal is equivalent to two independent binary PSK systems. Hence for coherent detection, the error probability is :

$$P_e = Q\left(\sqrt{2\gamma_b}\right), \ \gamma_b = \frac{\mathcal{E}_b}{N_0}, \ \mathcal{E}_b = \frac{1}{2}\int_0^T |u(t)|^2 \, dt$$

(c) Viterbi decoding (MLSE) of the MSK signal yields identical performance to that of part (b).

(d) MSK is basically binary FSK with frequency separation of $\Delta f = 1/2T$. For this frequency separation the binary signals are orthogonal with coherent detection. Consequently, the error probability for symbol-by-symbol detection of the MSK signal yields an error probability of

$$P_e = Q\left(\sqrt{\gamma_b}\right)$$

which is 3dB poorer relative to the optimum Viterbi detection scheme.
For non-coherent detection of the MSK signal, the correlation coefficient for $\Delta f = 1/2T$ is :

$$|\rho| = \frac{\sin \pi/2}{\pi/2} = 0.637$$

From the results in Sec. 5-4-4 we observe that the performance of the non coherent detector method is about 4 dB worse than the coherent FSK detector. hence the loss is about 7 dB compared to the optimum demodulator for the MSK signal.

Problem 5.45 :

(a) For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + 1 - p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned}
P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\
&\stackrel{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\
&= (1 - p - p)^n = (1-2p)^n
\end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned} P_{\text{even}} + P_{\text{odd}} &= 1 \\ P_{\text{even}} - P_{\text{odd}} &= (1-2p)^n \end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1 - (1-2p)^n)$$

(b) Expanding the quantity $(1-2p)^n$, we obtain

$$(1-2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1 - 1 + n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

Problem 5.46 :

The overall probability of error is approximated by (see 5-5-2)

$$P(e) = KQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.61$$

Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2/2 = 15.7$

Problem 5.47 :

(a) The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \qquad D = 3 \times 0.3048 \text{ m}$$

we obtain :

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

(b) The effective radiated power is :

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

(c) The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995 \times 10^{-9} = -85.23 \text{ dB} \ = -55.23 \text{ dBm}$$

Note that :

$$\text{dBm} \ = 10 \log_{10}\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log_{10}(\text{power in Watts })$$

Problem 5.48 :

(a) The antenna gain for a parabolic antenna of diameter D is :

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \qquad \text{and} \qquad D = 1 \text{ m}$$

we obtain :

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

(b) The effective radiated power is :

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

(c) The received power is :

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB } = -67.20 \text{ dBm}$$

Problem 5.49 :

The wavelength of the transmitted signal is:

$$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}$$

The gain of the parabolic antenna is:

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.6 \left(\frac{\pi 10}{0.03}\right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is:

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4 \times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

Problem 5.50 :

(a) Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by :

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.5 \left(\frac{3.14159 \times 50}{\frac{3 \times 10^8}{2 \times 10^9}}\right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

(b) If $\frac{\mathcal{E}_b}{N_0} = 10 \text{ dB } = 10$, then

$$R = \frac{P_R}{N_0} \left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

Problem 5.51 :

The wavelength of the transmission is :

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain :

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \Longrightarrow 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation (see 5-5-6) :

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} \Longrightarrow P_T = \frac{P_R}{G_T G_R} \left(4\pi \frac{d}{\lambda}\right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

CHAPTER 6

Problem 6.1 :

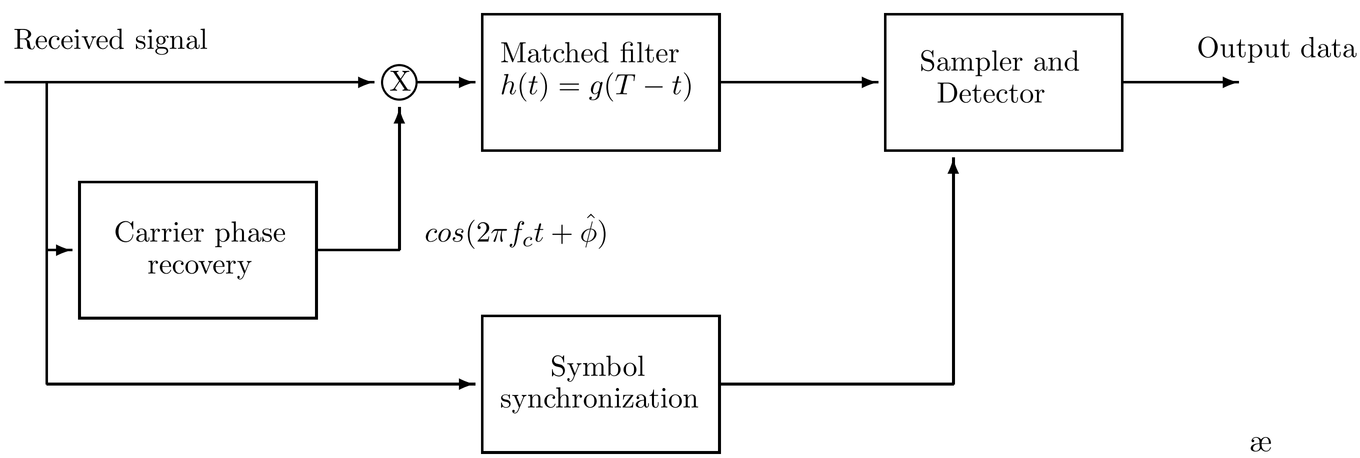
Using the relationship $r(t) = \sum_{n=1}^{N} r_n f_n(t)$ and $s(t;\psi) = \sum_{n=1}^{N} s_n(\psi) f_n(t)$ we have :

$$\begin{aligned}
\frac{1}{N_0}\int [r(t) - s(t;\psi)]^2 \, dt &= \frac{1}{N_0}\int \left[\sum_{n=1}^{N}(r_n - s_n(\psi)) f_n(t)\right]^2 dt \\
&= \frac{1}{N_0}\int \sum_{n=1}^{N}\sum_{m=1}^{N}(r_n - s_n(\psi))(r_m - s_m(\psi)) f_n(t) f_m(t) dt \\
&= \frac{1}{N_0}\sum_{n=1}^{N}\sum_{m=1}^{N}(r_n - s_n(\psi))(r_m - s_m(\psi))\delta_{mn} \\
&= \frac{1}{N_0}\sum_{n=1}^{N}(r_n - s_n(\psi))(r_n - s_n(\psi)) \\
&= \frac{1}{2\sigma^2}\sum_{n=1}^{N}[r_n - s_n(\psi)]^2
\end{aligned}$$

where we have exploited the orthonormality of the basis functions $f_n(t)$: $\int_{T_0} f_n(t) f_m(t) dt = \delta_{mn}$ and $\sigma^2 = \frac{N_0}{2}$.

Problem 6.2 :

A block diagram of a binary PSK receiver that employs match filtering is given in the following figure :



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As we note, the received signal is, first, multiplied with $cos(2\pi f_c t + \hat{\phi})$ and then fed the matched filter. This allows us to have the filter matched to the baseband pulse $g(t)$ and not to the passband signal.

If we want to have the filter matched to the passband signal, then the carrier phase estimate is fed into the matched filter, which should have an impulse response:

$$\begin{aligned} h(t) &= s(T-t) = g(T-t)cos(2\pi f_c(T-t) + \hat{\phi}) \\ &= g(T-t)[cos(2\pi f_c T)cos(-2\pi f_c t + \hat{\phi}) + sin(2\pi f_c T)sin(-2\pi f_c t + \hat{\phi}) \\ &= g(T-t)cos(-2\pi f_c t + \hat{\phi}) = g(T-t)cos(2\pi f_c t - \hat{\phi}) \end{aligned}$$

where we have assumed that $f_c T$ is an integer so that : $cos(2\pi f_c T) = 1,\ sin(2\pi f_c T) = 0$. As we note, in this case the impulse response of the filter should change according to the carrier phase estimate, something that is difficult to implement in practise. Hence, the initial realization (shown in the figure) is preferable.

Problem 6.3 :

(a) The closed loop transfer function is :

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

Since the real part of the roots is negative, the poles lie in the left half plane and therefore, the system is stable.

(b) Writing the denominator in the form :

$$D = s^2 + 2\zeta\omega_n s + \omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\zeta = \frac{1}{\sqrt{2}}$

Problem 6.4 :

(a) The closed loop transfer function is :

$$H(s) = \frac{G(s)/s}{1 + G(s)/s} = \frac{G(s)}{s + G(s)} = \frac{K}{\tau_1 s^2 + s + K} = \frac{\frac{K}{\tau_1}}{s^2 + \frac{1}{\tau_1}s + \frac{K}{\tau_1}}$$

The gain of the system at $f = 0$ is :

$$|H(0)| = |H(s)|_{s=0} = 1$$

(b) The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-1 \pm \sqrt{1 - 4K\tau_1}}{2\tau_1}$$

In order for the system to be stable the real part of the poles must be negative. Since K is greater than zero, the latter implies that τ_1 is positive. If in addition we require that the damping factor $\zeta = \frac{1}{2\sqrt{\tau_1 K}}$ is less than 1, then the gain K should satisfy the condition :

$$K > \frac{1}{4\tau_1}$$

Problem 6.5 :

The transfer function of the RC circuit is :

$$G(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

From the last equality we identify the time constants as :

$$\tau_2 = R_2 C, \qquad \tau_1 = (R_1 + R_2)C$$

Problem 6.6 :

Assuming that the input resistance of the operational amplifier is high so that no current flows through it, then the voltage-current equations of the circuit are :

$$\begin{aligned} V_2 &= -AV_1 \\ V_1 - V_2 &= \left(R_1 + \frac{1}{Cs}\right) i \\ V_1 - V_0 &= iR \end{aligned}$$

where, V_1, V_2 is the input and output voltage of the amplifier respectively, and V_0 is the signal at the input of the filter. Eliminating i and V_1, we obtain :

$$\frac{V_2}{V_1} = \frac{\frac{R_1 + \frac{1}{Cs}}{R}}{1 + \frac{1}{A} - \frac{R_1 + \frac{1}{Cs}}{AR}}$$

If we let $A \to \infty$ (ideal amplifier), then :

$$\frac{V_2}{V_1} = \frac{1 + R_1 Cs}{RCs} = \frac{1 + \tau_2 s}{\tau_1 s}$$

Hence, the constants τ_1, τ_2 of the active filter are given by :

$$\tau_1 = RC, \qquad \tau_2 = R_1 C$$

Problem 6.7 :

In the non decision-directed timing recovery method we maximize the function :

$$\Lambda_L(\tau) = \sum_m y_m^2(\tau)$$

with respect to τ. Thus, we obtain the condition :

$$\frac{d\Lambda_L(\tau)}{d\tau} = 2\sum_m y_m(\tau)\frac{dy_m(\tau)}{d\tau} = 0$$

Suppose now that we approximate the derivative of the log-likelihood $\Lambda_L(\tau)$ by the finite difference :

$$\frac{d\Lambda_L(\tau)}{d\tau} \approx \frac{\Lambda_L(\tau+\delta) - \Lambda_L(\tau-\delta)}{2\delta}$$

Then, if we substitute the expression of $\Lambda_L(\tau)$ in the previous approximation, we obtain :

$$\begin{aligned}
\frac{d\Lambda_L(\tau)}{d\tau} &= \frac{\sum_m y_m^2(\tau+\delta) - \sum_m y_m^2(\tau-\delta)}{2\delta} \\
&= \frac{1}{2\delta}\sum_m \left[\left(\int r(t)g(t-mT-\tau-\delta)dt\right)^2 - \left(\int r(t)g(t-mT-\tau+\delta)dt\right)^2\right]
\end{aligned}$$

where $g(-t)$ is the impulse response of the matched filter in the receiver. However, this is the expression of the early-late gate synchronizer, where the lowpass filter has been substituted by the summation operator. Thus, the early-late gate synchronizer is a close approximation to the timing recovery system.

Problem 6.8 :

An on-off keying signal is represented as :

$$\begin{array}{ll}
s_1(t) = A\cos(2\pi f_c t + \phi_c), & 0 \le t \le T \text{ (binary 1)} \\
s_2(t) = 0, & 0 \le t \le T \text{ (binary 0)}
\end{array}$$

Let $r(t)$ be the received signal, that is $r(t) = s(t;\phi_c) + n(t)$ where $s(t;\phi_c)$ is either $s_1(t)$ or $s_2(t)$ and $n(t)$ is white Gaussian noise with variance $\frac{N_0}{2}$. The likelihood function, that is to be maximized with respect to ϕ_c over the inteval $[0,T]$, is proportional to :

$$\Lambda(\phi_c) = \exp\left[-\frac{2}{N_0}\int_0^T [r(t) - s(t;\phi_c)]^2 dt\right]$$

Maximization of $\Lambda(\phi_c)$ is equivalent to the maximization of the log-likelihood function :

$$\begin{aligned}\Lambda_L(\phi_c) &= -\frac{2}{N_0}\int_0^T [r(t)-s(t;\phi_c)]^2 dt \\ &= -\frac{2}{N_0}\int_0^T r^2(t)dt + \frac{4}{N_0}\int_0^T r(t)s(t;\phi_c)dt - \frac{2}{N_0}\int_0^T s^2(t;\phi_c)dt\end{aligned}$$

Since the first term does not involve the parameter of interest ϕ_c and the last term is simply a constant equal to the signal energy of the signal over $[0,T]$ which is independent of the carrier phase, we can carry the maximization over the function :

$$V(\phi_c) = \int_0^T r(t)s(t;\phi_c)dt$$

Note that $s(t;\phi_c)$ can take two different values, $s_1(t)$ and $s_2(t)$, depending on the transmission of a binary 1 or 0. Thus, a more appropriate function to maximize is the average log-likelihood

$$\bar{V}(\phi_c) = \frac{1}{2}\int_0^T r(t)s_1(t)dt + \frac{1}{2}\int_0^T r(t)s_2(t)dt$$

Since $s_2(t) = 0$, the function $\bar{V}(\phi_c)$ takes the form :

$$\bar{V}(\phi_c) = \frac{1}{2}\int_0^T r(t)A\cos(2\pi f_c t + \phi_c)dt$$

Setting the derivative of $\bar{V}(\phi_c)$ with respect to ϕ_c equal to zero, we obtain :

$$\begin{aligned}\frac{\vartheta \bar{V}(\phi_c)}{\vartheta \phi_c} = 0 &= \frac{1}{2}\int_0^T r(t)A\sin(2\pi f_c t + \phi_c)dt \\ &= \cos\phi_c \frac{1}{2}\int_0^T r(t)A\sin(2\pi f_c t)dt + \sin\phi_c \frac{1}{2}\int_0^T r(t)A\cos(2\pi f_c t)dt\end{aligned}$$

Thus, the maximum likelihood estimate of the carrier phase is :

$$\hat{\phi}_{c,ML} = -\arctan\left[\frac{\int_0^T r(t)\sin(2\pi f_c t)dt}{\int_0^T r(t)\cos(2\pi f_c t)dt}\right]$$

Problem 6.9 :

(a) The wavelength λ is :

$$\lambda = \frac{3\times 10^8}{10^9}\ \text{m} = \frac{3}{10}\ \text{m}$$

Hence, the Doppler frequency shift is :

$$f_D = \pm\frac{u}{\lambda} = \pm\frac{100\ \text{Km/hr}}{\frac{3}{10}\ \text{m}} = \pm\frac{100\times 10^3\times 10}{3\times 3600}\text{Hz} = \pm 92.5926\ \text{Hz}$$

The plus sign holds when the vehicle travels towards the transmitter whereas the minus sign holds when the vehicle moves away from the transmitter.

(b) The maximum difference in the Doppler frequency shift, when the vehicle travels at speed 100 km/hr and $f = 1$ GHz, is :

$$\Delta f_{D\max} = 2 f_D = 185.1852 \text{ Hz}$$

This should be the bandwith of the Doppler frequency tracking loop.

(c) The maximum Doppler frequency shift is obtained when $f = 1$ GHz $+ 1$ MHz and the vehicle moves towards the transmitter. In this case :

$$\lambda_{\min} = \frac{3 \times 10^8}{10^9 + 10^6} \text{ m } = 0.2997 \text{ m}$$

and therefore :

$$f_{D\max} = \frac{100 \times 10^3}{0.2997 \times 3600} = 92.6853 \text{ Hz}$$

Thus, the Doppler frequency spread is $B_d = 2 f_{D\max} = 185.3706$ Hz.

Problem 6.10 :

The maximum likelihood phase estimate given by (6-2-38) is :

$$\hat{\phi}_{ML} = -\tan^{-1} \frac{Im\left[\sum_{n=0}^{K-1} I_n^* y_n\right]}{Re\left[\sum_{n=0}^{K-1} I_n^* y_n\right]}$$

where $y_n = \int_{nT}^{(n+1)T} r(t) g^*(t - nT) dt$. The $Re(y_n)$, $Im(y_n)$ are statistically independent components of y_n. Since $r(t) = e^{-j\phi} \sum_n I_n g(t - nT) + z(t)$ it follows that $y_n = I_n e^{-j\phi} + z_n$, where the pulse energy is normalized to unity. Then :

$$\sum_{n=0}^{K-1} I_n^* y_n = \sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right]$$

Hence :

$$E\left\{ Im\left[\sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right] \right] \right\} = -K \left| \bar{I}_n \right|^2 \sin\phi$$

and

$$E\left\{ Re\left[\sum_{n=0}^{K-1} \left[|I_n|^2 e^{-j\phi} + I_n^* z_n \right] \right] \right\} = -K \left| \bar{I}_n \right|^2 \cos\phi$$

Consequently : $E\left[\hat{\phi}_{ML}\right] = -\tan^{-1}\frac{-\sin\phi}{\cos\phi} = \phi$, and hence, $\hat{\phi}_{ML}$ is an unbiased estimate of the true phase ϕ.

Problem 6.11 :

The procedure that is used in Sec. 5-2-7 to derive the pdf $p(\Theta_r)$ for the phase of a PSK signal may be used to determine the pdf $p(\hat{\phi}_{ML})$. Specifically, we have :

$$\hat{\phi}_{ML} = -\tan^{-1}\frac{Im\left[\sum_{n=0}^{K-1} I_n^* y_n\right]}{Re\left[\sum_{n=0}^{K-1} I_n^* y_n\right]}$$

where $y_n = \int_{nT}^{(n+1)T} r(t)g^*(t-nT)dt$ and $r(t) = e^{-j\phi}\sum_n I_n g(t-nT) + z(t)$. Substitution of $r(t)$ into y_n yields : $y_n = I_n e^{-j\phi} + z_n$. Hence :

$$\sum_{n=0}^{K-1} I_n^* y_n = e^{-j\phi}\sum_{n=0}^{K-1} |I_n|^2 + \sum_{n=0}^{K-1} I_n^* z_n$$

$$U + jV = Ce^{-j\phi} + z = C\cos\phi + x + j(y - C\sin\phi)$$

where $C = \sum_{n=0}^{K-1} |I_n|^2$ and $z = \sum_{n=0}^{K-1} I_n^* z_n = x + jy$. The random variables (x,y) are zero-mean, Gaussian random variables with variances σ^2.Hence :

$$p(U,V) = \frac{1}{2\pi\sigma^2}e^{-\left[(U-C\cos\phi)^2-(V-C\sin\phi)^2\right]}$$

By defining $R = \sqrt{U^2+V^2}$ and $\hat{\phi}_{ML} = \tan^{-1}\frac{V}{U}$ and making the change in variables, we obtain $p(R,\hat{\phi}_{ML})$ and finally, $p(\hat{\phi}_{ML}) = \int_0^\infty p(r,\hat{\phi}_{ML})dr$. Upon performing the integration over R, as in Sec. 5-2-7, we obtain :

$$p(\hat{\phi}_{ML}) = \frac{1}{2\pi}e^{-2\gamma\sin^2\hat{\phi}_{ML}}\int_0^\infty re^{-\left(r-\sqrt{4\gamma}\cos\hat{\phi}_{ML}\right)^2/2}dr$$

where $\gamma = C^2/2\sigma^2$. The graph of $p(\hat{\phi}_{ML})$ is identical to that given on page 271, Fig. 5-2-9. We observe that $E(\hat{\phi}_{ML}) = \phi$, so that the estimate is unbiased.

Problem 6.12 :

We begin with the log-likelihood function given in (6-2-35), namely :

$$\Lambda_L(\phi) = Re\left\{\left[\frac{1}{N_0}\int_{T_0} r(t)s_l^*(t)dt\right]e^{j\phi}\right\}$$

where $s_l(t)$ is given as : $s_l(t) = \sum_n I_n g(t-nT) + j\sum_n J_n u(t-nT-T/2)$. Again we define $y_n = \int_{nT}^{(n+1)T} r(t)s_l^*(t-nT)dt$. Also, let : $x_n = \int_{(n+1/2)T}^{(n+3/2)T} r(t)s_l^*(t-nT-T/2)dt$. Then :

$$\begin{array}{rcl}\Lambda_L(\phi) & = & Re\left\{\frac{e^{j\phi}}{N_0}\left[\sum_{n=0}^{K-1} I_n^* y_n - j\sum_{n=0}^{K-1} J_n^* x_n\right]\right\} \\ & = & Re\left[A\cos\phi + jA\sin\phi\right]\end{array}$$

where $A = \sum_{n=0}^{K-1} I_n^* y_n - j\sum_{n=0}^{K-1} J_n^* x_n$. Thus : $\Lambda_L(\phi) = Re(A)\cos\phi - Im(A)\sin\phi$ and :

$$\frac{d\Lambda_L(\phi)}{d\phi} = -Re(A)\sin\phi - Im(A)\cos\phi = 0 \Rightarrow$$

$$\hat{\phi}_{ML} = -\tan^{-1}\frac{Im\left[\sum_{n=0}^{K-1} I_n^* y_n - j\sum_{n=0}^{K-1} J_n^* x_n\right]}{Re\left[\sum_{n=0}^{K-1} I_n^* y_n - j\sum_{n=0}^{K-1} J_n^* x_n\right]}$$

Problem 6.13 :

Assume that the signal $u_m(t)$ is the input to the Costas loop. Then $u_m(t)$ is multiplied by $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$, where $\cos(2\pi f_c t + \hat{\phi})$ is the output of the VCO. Hence :

$$\begin{array}{rl} u_{mc}(t) & \\ = & A_m g_T(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \hat{\phi}) - A_m \hat{g}_T(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \hat{\phi}) \\ = & \frac{A_m g_T(t)}{2}\left[\cos(2\pi 2 f_c t + \hat{\phi}) + \cos(\hat{\phi})\right] - \frac{A_m \hat{g}_T(t)}{2}\left[\sin(2\pi 2 f_c t + \hat{\phi}) - \sin(\hat{\phi})\right] \\ u_{ms}(t) & \\ = & A_m g_T(t)\cos(2\pi f_c t)\sin(2\pi f_c t + \hat{\phi}) - A_m \hat{g}_T(t)\sin(2\pi f_c t)\sin(2\pi f_c t + \hat{\phi}) \\ = & \frac{A_m g_T(t)}{2}\left[\sin(2\pi 2 f_c t + \hat{\phi}) + \sin(\hat{\phi})\right] - \frac{A_m \hat{g}_T(t)}{2}\left[\cos(\hat{\phi}) - \cos(2\pi 2 f_c t + \hat{\phi})\right] \end{array}$$

The lowpass filters of the Costas loop will reject the double frequency components, so that :

$$\begin{array}{rcl} y_{mc}(t) & = & \frac{A_m g_T(t)}{2}\cos(\hat{\phi}) + \frac{A_m \hat{g}_T(t)}{2}\sin(\hat{\phi}) \\ y_{ms}(t) & = & \frac{A_m g_T(t)}{2}\sin(\hat{\phi}) - \frac{A_m \hat{g}_T(t)}{2}\cos(\hat{\phi}) \end{array}$$

Note that when the carrier phase has been extracted correctly, $\hat{\phi} = 0$ and therefore :

$$y_{mc}(t) = \frac{A_m g_T(t)}{2}, \qquad y_{ms}(t) = -\frac{A_m \hat{g}_T(t)}{2}$$

If the second signal, $y_{ms}(t)$ is passed through a Hilbert transformer, then :

$$\hat{y}_{ms}(t) = -\frac{A_m \hat{\hat{g}}_T(t)}{2} = \frac{A_m g_T(t)}{2}$$

and by adding this signal to $y_{mc}(t)$ we obtain the original unmodulated signal.

Problem 6.14 :

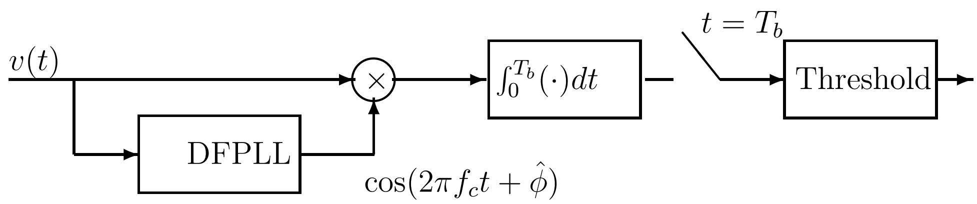
(a) The signal $r(t)$ can be written as :

$$\begin{aligned} r(t) &= \pm\sqrt{2P_s}\cos(2\pi f_c t+\phi)+\sqrt{2P_c}\sin(2\pi f_c t+\phi) \\ &= \sqrt{2(P_c+P_s)}\sin\left(2\pi f_c t+\phi+a_n\tan^{-1}\left(\sqrt{\frac{P_s}{P_c}}\right)\right) \\ &= \sqrt{2P_T}\sin\left(2\pi f_c t+\phi+a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right) \end{aligned}$$

where $a_n = \pm 1$ are the information symbols and P_T is the total transmitted power. As it is observed the signal has the form of a PM signal where :

$$\theta_n = a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)$$

Any method used to extract the carrier phase from the received signal can be employed at the receiver. The following figure shows the structure of a receiver that employs a decision-feedback PLL. The operation of the PLL is described in the next part.



(b) At the receiver (DFPLL) the signal is demodulated by crosscorrelating the received signal :

$$r(t) = \sqrt{2P_T}\sin\left(2\pi f_c t+\phi+a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right)+n(t)$$

with $\cos(2\pi f_c t+\hat{\phi})$ and $\sin(2\pi f_c t+\hat{\phi})$. The sampled values at the ouput of the correlators are :

$$\begin{aligned} r_1 &= \frac{1}{2}\left[\sqrt{2P_T}-n_s(t)\right]\sin(\phi-\hat{\phi}+\theta_n)+\frac{1}{2}n_c(t)\cos(\phi-\hat{\phi}+\theta_n) \\ r_2 &= \frac{1}{2}\left[\sqrt{2P_T}-n_s(t)\right]\cos(\phi-\hat{\phi}+\theta_n)+\frac{1}{2}n_c(t)\sin(\hat{\phi}-\phi-\theta_n) \end{aligned}$$

where $n_c(t)$, $n_s(t)$ are the in-phase and quadrature components of the noise $n(t)$. If the detector has made the correct decision on the transmitted point, then by multiplying r_1 by $\cos(\theta_n)$ and r_2 by $\sin(\theta_n)$ and subtracting the results, we obtain (after ignoring the noise) :

$$\begin{aligned}
r_1\cos(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\sin(\phi-\hat{\phi})\cos^2(\theta_n)+\cos(\phi-\hat{\phi})\sin(\theta_n)\cos(\theta_n)\right]\\
r_2\sin(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\cos(\phi-\hat{\phi})\cos(\theta_n)\sin(\theta_n)-\sin(\phi-\hat{\phi})\sin^2(\theta_n)\right]\\
e(t) &= r_1\cos(\theta_n)-r_2\sin(\theta_n)=\frac{1}{2}\sqrt{2P_T}\sin(\phi-\hat{\phi})
\end{aligned}$$

The error $e(t)$ is passed to the loop filter of the DFPLL that drives the VCO. As it is seen only the phase θ_n is used to estimate the carrier phase.

(c) Having a correct carrier phase estimate, the output of the lowpass filter sampled at $t = T_b$ is :

$$\begin{aligned}
r &= \pm\frac{1}{2}\sqrt{2P_T}\sin\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)+n\\
&= \pm\frac{1}{2}\sqrt{2P_T}\sqrt{1-\frac{P_c}{P_T}}+n\\
&= \pm\frac{1}{2}\sqrt{2P_T\left(1-\frac{P_c}{P_T}\right)}+n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance :

$$\begin{aligned}
\sigma_n^2 &= E\left[\int_0^{T_b}\int_0^{T_b} n(t)n(\tau)\cos(2\pi f_c t+\phi)\cos(2\pi f_c\tau+\phi)dtd\tau\right]\\
&= \frac{N_0}{2}\int_0^{T_b}\cos^2(2\pi f_c t+\phi)dt\\
&= \frac{N_0}{4}
\end{aligned}$$

Note that T_b has been normalized to 1 since the problem has been stated in terms of the power of the involved signals. The probability of error is given by :

$$P(\text{error}) = Q\left[\sqrt{\frac{2P_T}{N_0}\left(1-\frac{P_c}{P_T}\right)}\right]$$

The loss due to the allocation of power to the pilot signal is :

$$\text{SNR}_{\text{loss}} = 10\log_{10}\left(1-\frac{P_c}{P_T}\right)$$

When $P_c/P_T = 0.1$, then $\text{SNR}_{\text{loss}} = 10\log_{10}(0.9) = -0.4576$ dB. The negative sign indicates that the SNR is decreased by 0.4576 dB.

Problem 6.15 :

The received signal-plus-noise vector at the output of the matched filter may be represented as (see (5-2-63) for example) :

$$r_n = \sqrt{\mathcal{E}_s} e^{j(\theta_n - \phi)} + N_n$$

where $\theta_n = 0, \pi/2, \pi, 3\pi/2$ for QPSK, and ϕ is the carrier phase. By raising r_n to the fourth power and neglecting all products of noise terms, we obtain :

$$\begin{aligned} r_n^4 &\approx \left(\sqrt{\mathcal{E}_s}\right)^4 e^{j4(\theta_n - \phi)} + 4\left(\sqrt{\mathcal{E}_s}\right)^3 N_n \\ &\approx \left(\sqrt{\mathcal{E}_s}\right)^3 \left[\sqrt{\mathcal{E}_s} e^{-j4\phi} + 4N_n\right] \end{aligned}$$

If the estimate is formed by averaging the received vectors $\{r_n^4\}$ over K signal intervals, we have the resultant vector $U = K\sqrt{\mathcal{E}_s} e^{-j\phi} + 4\sum_{n=1}^K N_n$. Let $\phi_4 \equiv 4\phi$. Then, the estimate of ϕ_4 is :

$$\hat{\phi}_4 = -\tan^{-1}\frac{Im(U)}{Re(U)}$$

N_n is a complex-valued Gaussian noise component with zero mean and variance $\sigma^2 = N_0/2$. Hence, the pdf of $\hat{\phi}_4$ is given by (5-2-55) where :

$$\gamma_s = \frac{\left(K\sqrt{\mathcal{E}_s}\right)^2}{16\left(2K\sigma^2\right)} = \frac{K^2\mathcal{E}_s}{16KN_0} = \frac{K\mathcal{E}_s}{16N_0}$$

To a first approximation, the variance of the estimate is :

$$\sigma^2_{\hat{\phi}_4} \approx \frac{1}{\gamma_s} = \frac{16}{K\mathcal{E}_s/N_0}$$

Problem 6.16 :

The PDF of the carrier phase error ϕ_e, is given by :

$$p(\phi_e) = \frac{1}{\sqrt{2\pi}\sigma_\phi} e^{-\frac{\phi_e^2}{2\sigma_\phi^2}}$$

Thus the average probability of error is :

$$\begin{aligned} \bar{P}_2 &= \int_{-\infty}^{\infty} P_2(\phi_e) p(\phi_e) d\phi_e \\ &= \int_{-\infty}^{\infty} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}\cos^2\phi_e}\right] p(\phi_e) d\phi_e \\ &= \frac{1}{2\pi\sigma_\phi}\int_{-\infty}^{\infty}\int_{\sqrt{\frac{2\mathcal{E}_b}{N_0}\cos^2\phi_e}}^{\infty} \exp\left[-\frac{1}{2}\left(x^2 + \frac{\phi_e^2}{\sigma_\phi^2}\right)\right] dx d\phi_e \end{aligned}$$

Problem 6.17:

The log-likelihood function of the symbol timing may be expressed in terms of the equivalent low-pass signals as

$$\begin{aligned} \Lambda_L(\tau) &= \Re\left[\frac{1}{N_0}\int_{T_0} r(t)s_l{}^*(t;\tau)dt\right] \\ &= \Re\left[\frac{1}{N_0}\int_{T_0} r(t)\sum_n I_n{}^*g^*(t-nT-\tau)dt\right] \\ &= \Re\left[\frac{1}{N_0}\sum_n I_n{}^*y_n(\tau)\right] \end{aligned}$$

where $y_n(\tau) = \int_{T_0} r(t)g^*(t-nT-\tau)dt.$
A necessary condition for $\hat{\tau}$ to be the ML estimate of τ is

$$\begin{aligned} \frac{d\Lambda_L(\tau)}{\tau} &= 0 \Rightarrow \\ \frac{d}{d\tau}\left[\sum_n I_n{}^*y_n(\tau) + \sum_n I_n y_n{}^*(\tau)\right] &= 0 \Rightarrow \\ \sum_n I_n{}^*\frac{d}{d\tau}y_n(\tau) + \sum_n I_n\frac{d}{d\tau}y_n{}^*(\tau) &= 0 \end{aligned}$$

If we express $y_n(\tau)$ into its real and imaginary parts : $y_n(\tau) = a_n(\tau) + jb_n(\tau)$, the above expression simplifies to the following condition for the ML estimate of the timing $\hat{\tau}$

$$\sum_n \Re[I_n]\frac{d}{d\tau}a_n(\tau) + \sum_n \Im[I_n]\frac{d}{d\tau}b_n(\tau) = 0$$

Problem 6.18:

We follow the exact same steps of the derivation found in Sec. 6.4. For a PAM signal $I_n{}^* = I_n$ and $J_n = 0$. Since the pulse $g(t)$ is real, it follows that $B(\tau)$ in expression (6.4-6) is zero, therefore (6.4-7) can be rewritten as

$$\Lambda_L(\phi,\tau) = A(\tau)\cos\phi$$

where

$$A(\tau) = \frac{1}{N_0}\sum I_n y_n(\tau)$$

Then the necessary conditions for the estimates of ϕ and τ to be the ML estimates (6.4-8) and (6.4-9) give

$$\hat{\phi}_{ML} = 0$$

and

$$\sum_n I_n \frac{d}{d\tau}[y_n(\tau)]_{\tau=\hat{\tau}_{ML}} = 0$$

Problem 6.19:

The derivation for the ML estimates of ϕ and τ for an offset QPSK signal follow the derivation found in Sec. 6.4, with the added simplification that, since $w(t) = g(t - T/2)$, we have that $x_n(\tau) = y_n(\tau + T/2)$.

CHAPTER 7

Problem 7.1 :

$$I(x_j;Y) = \sum_{i=0}^{Q-1} P(y_i|x_j) \log \frac{P(y_i|x_j)}{P(y_i)}$$

Since

$$\sum_{j=0}^{q-1} P(x_j) = \sum_{P(x_j)\neq 0} P(X_j) = 1$$

we have :

$$\begin{array}{rcl} I(X;Y) & = & \sum_{j=0}^{q-1} P(x_j) I(x_j;Y) = \sum_{P(x_j)\neq 0} C P(x_j) \\ & = & C \sum_{P(x_j)\neq 0} P(x_j) = C = \max_{P(x_j)} I(X;Y) \end{array}$$

Thus, the given set of $P(x_j)$ maximizes $I(X;Y)$ and the condition is sufficient.
To prove that the condition is necessary, we proceed as follows : Since $P(x_j)$ satisfies the condition $\sum_{j=0}^{q-1} P(x_j) = 1$, we form the cost function :

$$C(X) = I(X;Y) - \lambda \left[\sum_{j=0}^{q-1} P(x_j) - 1 \right]$$

and use the Lagrange multiplier method to maximize $C(X)$. The partial derivative of $C(X)$ with respect to all $P(X_j)$ is :

$$\begin{array}{rcl} \frac{\partial C(X)}{\partial P(x_k)} & = & \frac{\partial}{\partial P(x_k)} \left[\sum_{j=0}^{q-1} P(x_j) I(x_j;Y) - \lambda \sum_{j=0}^{q-1} P(x_j) + \lambda \right] \\ & = & I(x_k;Y) + \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) - \lambda = 0 \end{array}$$

But :

$$\begin{array}{rcl} \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) & = & -\log e \sum_{j=0}^{q-1} P(x_j) \sum_{i=0}^{Q-1} P(y_i|x_j) \frac{P(y_i)}{P(y_i|x_j)} \frac{P(y_i|x_j)}{-[P(y_i)]^2} \frac{\partial P(y_i)}{\partial P(x_k)} \\ & = & -\log e \sum_{i=0}^{Q-1} \left[\sum_{j=0}^{q-1} \frac{P(x_j)P(y_i|x_j)}{P(y_i)} \right] P(y_i|x_k) \\ & = & -\log e \sum_{i=0}^{Q-1} \frac{P(y_i)}{P(y_i)} P(y_i|x_k) = -\log e \end{array}$$

Therefore:

$$I(x_k;Y) + \sum_{j=0}^{q-1} P(x_j) \frac{\partial}{\partial P(x_k)} I(x_j;Y) - \lambda = 0 \Rightarrow I(x_k;Y) = \lambda + \log e, \quad \forall\ x_k$$

Now, we consider two cases :

(i) If there exists a set of $P(x_k)$ such that :

$$\sum_{k=0}^{q-1} P(x_k) = 1,\ 0 \leq P(x_k) \leq 1,\ k = 0, 1, ..., q-1 \text{ and } \frac{\partial C(X)}{\partial P(x_k)} = 0$$

then this set will maximize $I(X;Y)$, since $I(X;Y)$ is a convex function of $P(x_k)$. We have :

$$\begin{aligned} C &= \max_{P(x_j)} I(X;Y) = \sum_{j=0}^{q-1} P(x_j) I(x_j;Y) \\ &= \sum_{j=0}^{q-1} P(x_j)\left[\lambda + \log e\right] = \lambda + \log e = I(x_j;Y) \end{aligned}$$

This provides a sufficient condition for finding a set of $P(x_k)$ which maximizes $I(X;Y)$, for a symmetric memoryless channel.

(ii) If the set $\{P(x_k)\}$ does not satisfy $0 \leq P(x_k) \leq 1,\ k = 0, 1, ..., q-1$, since $I(X;Y)$ is a convex function of $P(x_k)$, necessary and sufficient conditions on $P(x_k)$ for maximizing $I(X;Y)$ are :

$$\frac{\partial I(X;Y)}{\partial P(x_k)} = \mu,\ \text{for } P(x_k) > 0,\ \frac{\partial I(X;Y)}{\partial P(x_k)} \leq \mu,\ \text{for } P(x_k) = 0$$

Hence :

$$\begin{aligned} I(x_k;Y) = \mu + \log e,\ P(x_k) \neq 0 \\ I(x_k;Y) \leq \mu + \log e,\ P(x_k) = 0 \end{aligned}$$

is the necessary and sufficient condition on $P(x_k)$ for maximizing $I(X;Y)$ and $\mu + \log e = C$.

Problem 7.2 :

(a) For a set of equally probable inputs with probability $1/M$, we have :

$$\begin{aligned} I(x_k;Y) &= \sum_{i=0}^{M-1} P(y_i|x_k) \log \tfrac{P(y_i|x_k)}{P(y_i)} \\ &= P(y_k|x_k) \log \tfrac{P(y_k|x_k)}{P(y_k)} + \sum_{i \neq k} P(y_i|x_k) \log \tfrac{P(y_i|x_k)}{P(y_i)} \end{aligned}$$

But $\forall i$:

$$P(y_i) = \sum_{j=0}^{M-1} P(y_i|x_j) = \frac{1}{M} P(y_i|x_i) + \frac{1}{M} \sum_{j \neq i} P(y_i|x_j) = \frac{1}{M}\left(1 - p + (M-1)\frac{p}{M-1}\right) = \frac{1}{M}$$

Hence :

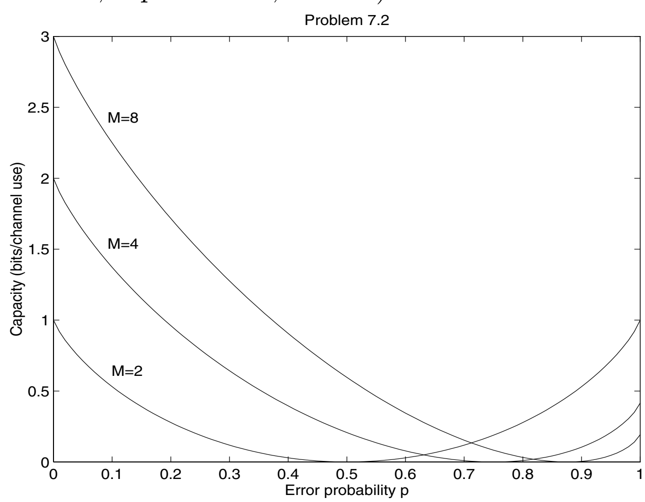
$$\begin{aligned} I(x_k;Y) &= (1-p) \log \tfrac{(1-p)}{1/M} + (M-1) \tfrac{p}{M-1} \log \tfrac{p/M-1}{1/M} \\ &= (1-p) \log\left(M(1-p)\right) + p \log\left(\tfrac{pM}{M-1}\right) \\ &= \log M + (1-p)\log(1-p) + p \log\left(\tfrac{p}{M-1}\right) \end{aligned}$$

which is the same for all $k = 0, 1, ...M-1$. Hence, this set of $\{P(x_k) = 1/M\}$ satisfies the condition of Probl. 7.1.

(b) From part (a) :

$$C = \log M + (1-p)\log(1-p) + p\log\left(\frac{p}{M-1}\right)$$

A plot of the capacity is given in the following figure. We note that the capacity of the channel is zero when all transitions are equally likely (i.e. when $1-p = \frac{p}{M-1} \Rightarrow p = \frac{M-1}{M}$ or: $p = 0.5,\ M=2;\quad p = 0.75, M=4;\quad p = 0.875, M=8$).



Problem 7.3 :

In all of these channels, we assume a set of $\{P(x_i)\}$ which, we think, may give the maximum $I(X;Y)$ and see if this set of $\{P(x_i)\}$ satisfies the condition of Probl. 7.1(or relationship 7-1-21). Since all the channels exhibit symmetries, the set of $\{P(x_i)\}$ that we examine first, is the equiprobable distribution.

(a) Suppose $P(x_i) = 1/4, \forall i$. Then : $P(y_1) = P(y_1|x_1)P(x_1) + P(y_1|x_4)P(x_4) = 1/4$. Similarly $P(y_j) = 1/4, \forall j$. Hence :

$$I(x_1;Y) = \sum_{j=1}^{4} P(y_j|x_1)\log\frac{P(y_j|x_1)}{P(y_j)} = \frac{1}{2}\log\frac{1/2}{1/4} + \frac{1}{2}\log\frac{1/2}{1/4} = \log 2 = 1$$

Similarly : $I(x_i;Y) = 1, i = 2,3,4$. Hence this set of input probabilities satisfies the condition of Probl. 7.1 and :

$$C = 1 \text{ bit/symbol sent (bit/channel use)}$$

(b) We assume that $P(x_i) = 1/2,\ i = 1, 2$. Then $P(y_1) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{1}{4}$ and similarly $P(y_j) = 1/4,\ j = 2, 3, 4$. Hence :

$$I(x_1; Y) = \sum_{j=1}^{4} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 2\frac{1}{3} \log \frac{1/3}{1/4} + 2\frac{1}{6} \log \frac{1/6}{1/4} = 0.0817$$

and the same is true for $I(x_2; Y)$. Thus :

$$C = 0.0817 \text{ bits/symbol sent}$$

(c) We assume that $P(x_i) = 1/3,\ i = 1, 2, 3$. Then $P(y_1) = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{3}$ and similarly $P(y_j) = 1/3,\ j = 2, 3$. Hence :

$$I(x_1; Y) = \sum_{j=1}^{3} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = \frac{1}{2} \log \frac{1/2}{1/3} + \frac{1}{3} \log \frac{1/3}{1/3} + \frac{1}{6} \log \frac{1/6}{1/3} = 0.1258$$

and the same is true for $I(x_2; Y), I(x_3; Y)$. Thus :

$$C = 0.1258 \text{ bits/symbol sent}$$

Problem 7.4 :

We expect the first channel (with exhibits a certain symmetry) to achieve its capacity through equiprobable input symbols; the second one not.

(a) We assume that $P(x_i) = 1/2,\ i = 1, 2$. Then $P(y_1) = \frac{1}{2}(0.6 + 0.1) = 0.35$ and similarly $P(y_3) = 0.35, .\ P(y_2) = 0.3$. Hence :

$$I(x_1; Y) = \sum_{j=1}^{3} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 0.6 \log \frac{0.6}{0.35} + 0.3 \log \frac{0.3}{0.3} + 0.1 \log \frac{0.1}{0.35} = 0.2858$$

and the same is true for $I(x_2; Y)$. Thus, equally likely input symbols maximize the information rate through the channel and :

$$C = 0.2858 \text{ bits/symbol sent}$$

(b) We assume that $P(x_i) = 1/2,\ i = 1, 2$. Then $P(y_1) = \frac{1}{2}(0.6 + 0.3) = 0.45$, and similarly $P(y_2) = 0.2, P(y_3) = 0.35$. Hence :

$$I(x_1; Y) = \sum_{j=1}^{3} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 0.6 \log \frac{0.6}{0.45} + 0.3 \log \frac{0.3}{0.2} + 0.1 \log \frac{0.1}{0.35} = 0.244$$

But :

$$I(x_2;Y) = \sum_{j=1}^{3} P(y_j|x_2) \log \frac{P(y_j|x_2)}{P(y_j)} = 0.3 \log \frac{0.3}{0.45} + 0.1 \log \frac{0.1}{0.2} + 0.6 \log \frac{0.6}{0.35} = 0.191$$

Since $I(x_1;Y) \neq I(x_2;Y)$ the equiprobable input distribution does not maximize the information rate through the channel. To determine $P(x_1), P(x_2)$ that give the channel capacity, we assume that $P(x_1) = a, P(x_2) = 1 - a$; then we find $P(y_i)$, express $I(X;Y)$ as a function of a and set its derivative with respect to a equal to zero.

Problem 7.5 :

(a) Relationship (7-1-31) gives :

$$C = W \log \left(1 + \frac{P_{av}}{WN_0}\right) = 25.9 \text{ Kbits/ sec}$$

(b) From Table 3-5-2 we see that logarithmic PCM uses 7-8 bits/sample and since we sample the speech signal at 8 KHz, this requires 56 to 64 Kbits/sec for speech transmission. Clearly, the above channel cannot support this transmission rate.

(c) The achievable transmission rate is :

$$0.7C = 18.2 \text{ Kbits/ sec}$$

From Table 3-5-2 we see that linear predictive coding (LPC) and adaptive delta modulation (ADM) are viable source coding methods for speech transmission over this channel.

Problem 7.6 :

(a) We assume that $P(x_i) = 1/2,\ i = 1, 2$. Then $P(y_1) = \frac{1}{2}\left(\frac{1}{2}(1-p) + \frac{1}{2}p\right) = \frac{1}{4}$ and similarly $P(y_j) = 1/4,\ j = 2, 3, 4$. Hence :

$$\begin{aligned} I(x_1;Y) &= \sum_{j=1}^{4} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = 2\frac{1}{2}(1-p) \log \frac{(1-p)/2}{1/4} + 2\frac{1}{2}p \log \frac{p/2}{1/4} \\ &= 1 + p \log p + (1-p) \log(1-p) \end{aligned}$$

and the same is true for $I(x_2;Y)$. Thus, equiprobable input symbols achieve the channel capacity

$$C = 1 + p \log p + (1-p) \log(1-p) \text{ bits/symbol sent}$$

(b) We note that the above capacity is the same to the capacity of the binary symmetric channel. Indeed, if we consider the grouping of the output symbols into $a = \{y_1, y_2\}$ and $b = \{y_3, y_4\}$ we get a binary symmetric channel, with transition probabilities: $P(a|x_1) = P(y_1|x_1) + P(y_2|x_1) = (1-p),\ P(a|x_2) = p$, etc.

Problem 7.7 :

We assume that $P(x_i) = 1/3,\ i = 1, 2, 3$. Then $P(y_1) = \frac{1}{3}((1-p)+p) = \frac{1}{3}$ and similarly $P(y_j) = 1/3,\ j = 2, 3$. Hence :

$$\begin{aligned} I(x_1;Y) &= \textstyle\sum_{j=1}^{3} P(y_j|x_1) \log \frac{P(y_j|x_1)}{P(y_j)} = (1-p)\log\frac{(1-p)}{1/3} + p\log\frac{p}{1/3} \\ &= \log 3 + p\log p + (1-p)\log(1-p) \end{aligned}$$

and the same is true for $I(x_2;Y),\ I(x_3;Y)$. Thus, equiprobable input symbols achieve the channel capacity :

$$C = \log 3 + p\log p + (1-p)\log(1-p) \text{ bits/symbol sent}$$

Problem 7.8 :

(a) the probability that a codeword transmitted over the BSC is received correctly, is equal to the probability that all R bits are received correctly. Since each bit transmission is independent from the others :

$$P(\text{correct codeword}) = (1-p)^R$$

(b)

$$P(\text{ at least one bit error in the codeword}) = 1 - P(\text{correct codeword}) = 1 - (1-p)^R$$

(c)

$$P(\text{ n}_\text{e} \text{ or less errors in R bits}) = \sum_{i=1}^{n_e} \binom{R}{i} p^i (1-p)^{R-i}$$

(d) For $R = 5,\ p = 0.01,\ n_e = 5$:

$$\begin{aligned} &(1-p)^R = 0.951 \\ &1-(1-p)^R = 0.049 \\ &\textstyle\sum_{i=1}^{n_e} \binom{R}{i} p^i (1-p)^{R-i} = 1-(1-p)^R = 0.049 \end{aligned}$$

Problem 7.9 :

Let $\mathbf{X} = (X_1, X_2, ..., X_n)$, $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)$. Since the channel is memoryless : $P(\mathbf{Y}|\mathbf{X}) = \prod_{i=1}^n P(Y_i|X_i)$ and :

$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X},\mathbf{Y}) \log \frac{P(\mathbf{Y}|\mathbf{X})}{P(\mathbf{Y})} \\ &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X},\mathbf{Y}) \log \frac{\prod_i P(Y_i|X_i)}{P(\mathbf{Y})} \end{aligned}$$

For statistically independent input symbols :

$$\begin{aligned} \sum_{i=1}^n I(X_i;Y_i) &= \sum_{i=1}^n \sum_{X_i} \sum_{Y_i} P(X_i,Y_i) \log \frac{P(Y_i|X_i)}{P(Y_i)} \\ &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X},\mathbf{Y}) \log \frac{\prod_i P(Y_i|X_i)}{\prod_i P(Y_i)} \end{aligned}$$

Then :

$$\begin{aligned} I(\mathbf{X};\mathbf{Y}) - \sum_{i=1}^n I(X_i;Y_i) &= \sum_{\mathbf{X}} \sum_{\mathbf{Y}} P(\mathbf{X},\mathbf{Y}) \log \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} \\ &= \sum_{\mathbf{Y}} P(\mathbf{Y}) \log \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} = \sum_{\mathbf{Y}} P(\mathbf{Y}) \ln \frac{\prod_i P(Y_i)}{P(\mathbf{Y})} \log e \\ &\leq \sum_{\mathbf{Y}} P(\mathbf{Y}) \left[\frac{\prod_i P(Y_i)}{P(\mathbf{Y})} - 1 \right] \log e \\ &= \left(\sum_{\mathbf{Y}} \prod_i P(Y_i) - \sum_{\mathbf{Y}} P(\mathbf{Y}) \right) \log e = (1-1)\log e = 0 \end{aligned}$$

where we have exploited the fact that : $\ln u \leq u - 1$, with equality iff $u = 1$. Therefore : $I(\mathbf{X};\mathbf{Y}) \leq \sum_{i=1}^n I(X_i;Y_i)$ with equality iff the set of input symbols is statistically independent.

Problem 7.10 :

$P(X = 0) = a$, $P(X = 1) = 1 - a$. Then : $P(Y = 0) = (1-p)a$, $P(Y = e) = p(1 - a + a) = p$, $P(Y = 1) = (1-p)(1-a)$.

(a)

$$\begin{aligned} I(X;Y) &= \sum_{i=1}^2 \sum_{j=1}^3 P(y_j|x_i)P(x_i) \log \frac{P(y_j|x_i)}{P(y_j)} \\ &= a(1-p)\log \frac{1-p}{a(1-p)} + ap \log \frac{p}{p} + (1-a)p \log \frac{p}{p} + (1-a)(1-p) \log \frac{1-p}{(1-a)(1-p)} \\ &= -(1-p)\left[a \log a + (1-a)\log(1-a)\right] \end{aligned}$$

Note that the term $-\left[a \log a + (1-a)\log(1-a)\right]$ is the entropy of the source.

(b) The value of a that maximizes $I(X;Y)$ is found from :

$$\frac{dI(X;Y)}{da} = 0 \Rightarrow \log a + \frac{a}{a}\log e - \log(1-a) - \frac{1-a}{1-a}\log e = 0 \Rightarrow a = 1/2$$

With this value of a, the resulting channel capacity is :

$$C = I(X;Y)|_{a=1/2} = 1 - p \text{ bits/channel use}$$

(c) $I(x;y) = \log \frac{P(y|x)}{P(y)}$. Hence :

$$I(0;0) = \log \tfrac{1-p}{(1-p)/2} = 1$$

$$I(1;1) = \log \tfrac{1-p}{(1-p)/2} = 1$$

$$I(0;e) = \log \tfrac{p}{p} = 0$$

$$I(1;e) = \log \tfrac{p}{p} = 0$$

Problem 7.11 :

(a) The cutoff rate for the binary input, ternary output channel is given by :

$$R_3 = \max_{p_j} \left\{ -\log \sum_{i=0}^{2} \left[\sum_{j=0}^{1} P_j \sqrt{P(i|j)} \right]^2 \right\}$$

To maximize the term inside the brackets we want to minimize the argument S of the log function : $S = \sum_{i=0}^{2} \left[\sum_{j=0}^{1} P_j \sqrt{P(i|j)} \right]^2$. Suppose that $P_0 = x, P_1 = 1-x$. Then :

$$\begin{aligned} S &= \left(x\sqrt{1-p-a} + (1-x)\sqrt{p}\right)^2 + (x\sqrt{a} + (1-x)\sqrt{a})^2 + \left(x\sqrt{p} + (1-x)\sqrt{1-p-a}\right)^2 \\ &= 2\left(1-a-2\sqrt{p-p^2-ap}\right)x^2 - 2\left(1-a-2\sqrt{p-p^2-ap}\right)x + 1 \end{aligned}$$

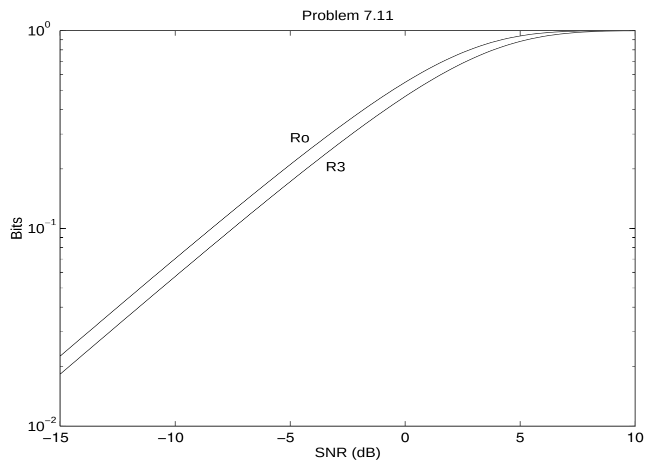
By setting : $\frac{dS}{dx} = 0$, we obtain $x = 1/2$ which corresponds to a minimum for S, since $\frac{d^2S}{dx^2}|_{x=1/2} > 0$. Then :

$$R_3 = -\log S = -\log \left\{ \frac{1 + a + 2\sqrt{p-p^2-ap}}{2} \right\} = 1 - \log\left(1 + a + 2\sqrt{p(1-p-a)}\right)$$

(b) For $\beta = 0.65\sqrt{N_0/2}$, we have :

$$\begin{array}{l} p = \frac{1}{\sqrt{\pi N_0}} \int_{\beta}^{\infty} \exp(-(x+\sqrt{E_c})^2/N_0) = Q\left[0.65 + \sqrt{2E_c/N_0}\right] \\ a = \frac{1}{\sqrt{\pi N_0}} \int_{-\beta}^{\beta} \exp(-(x+\sqrt{E_c})^2/N_0) = Q\left[\sqrt{2E_c/N_0} - 0.65\right] - Q\left[\sqrt{2E_c/N_0} + 0.65\right] \end{array}$$

The plot of R_3 is given in the following figure. In this figure, we have also included the plot of $R_\infty = \log \frac{2}{1+\exp(-\sqrt{E_c/N_0})}$. As we see the difference in performance between continuous-output (soft-decision-decoding , R_∞) and ternary output (R_3) is approximately 1 dB.



Problem 7.12 :

The overall channel is a binary symmetric channel with crossover probability p. To find p note that an error occurs if an odd number of channels produce an error. Thus :

$$p = \sum_{k=\text{odd}} \left(\begin{array}{c} n \\ k \end{array} \right) \epsilon^k (1-\epsilon)^{n-k}$$

Using the results of Problem 5.45, we find that :

$$p = \frac{1}{2}\left[1 - (1-2\epsilon)^2\right]$$

and therefore :

$$C = 1 - H(p)$$

If $n \to \infty$, then $(1-2\epsilon)^n \to 0$ and $p \to \frac{1}{2}$. In this case

$$C = \lim_{n\to\infty} C(n) = 1 - H(\frac{1}{2}) = 0$$

Problem 7.13 :

(a) The capacity of the channel is :

$$C_1 = \max_{P(x)}[H(Y) - H(Y|X)]$$

But, $H(Y|X) = 0$ and therefore, $C_1 = \max_{P(x)} H(Y) = 1$ which is achieved for $P(0) = P(1) = \frac{1}{2}$.

(b) Let q be the probability of the input symbol 0, and thus $(1-q)$ the probability of the input symbol 1. Then :

$$\begin{aligned} H(Y|X) &= \sum_x P(x)H(Y|X=x) \\ &= qH(Y|X=0) + (1-q)H(Y|X=1) \\ &= (1-q)H(Y|X=1) = (1-q)H(0.5) = (1-q) \end{aligned}$$

The probability mass function of the output symbols is :

$$\begin{aligned} P(Y=c) &= qP(Y=c|X=0) + (1-q)P(Y=c|X=1) \\ &= q + (1-q)0.5 = 0.5 + 0.5q \\ P(Y=d) &= (1-q)0.5 = 0.5 - 0.5q \end{aligned}$$

Hence :

$$C_2 = \max_q[H(0.5 + 0.5q) - (1-q)]$$

To find the probability q that achieves the maximum, we set the derivative of C_2 with respect to q equal to 0. Thus,

$$\begin{aligned} \frac{\vartheta C_2}{\vartheta q} = 0 &= 1 - [0.5\log_2(0.5+0.5q) + (0.5+0.5q)\frac{0.5}{0.5+0.5q}\frac{1}{\ln 2}] \\ &\quad -[-0.5\log_2(0.5-0.5q) + (0.5-0.5q)\frac{-0.5}{0.5-0.5q}\frac{1}{\ln 2}] \\ &= 1 + 0.5\log_2(0.5-0.5q) - 0.5\log_2(0.5+0.5q) \end{aligned}$$

Therefore :

$$\log_2\frac{0.5-0.5q}{0.5+0.5q} = -2 \Longrightarrow q = \frac{3}{5}$$

and the channel capacity is :

$$C_2 = H(\frac{1}{5}) - \frac{2}{5} = 0.3219$$

(c) The transition probability matrix of the third channel can be written as :

$$\mathbf{Q} = \frac{1}{2}\mathbf{Q}_1 + \frac{1}{2}\mathbf{Q}_2$$

where $\mathbf{Q}_1$, $\mathbf{Q}_2$ are the transition probability matrices of channel 1 and channel 2 respectively. We have assumed that the output space of both channels has been augmented by adding two new symbols so that the size of the matrices $\mathbf{Q}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is the same. The transition probabilities to these newly added output symbols is equal to zero. Using the fact that the function $I(\mathbf{p};\mathbf{Q})$ is a convex function in $\mathbf{Q}$ we obtain :

$$\begin{aligned} C &= \max_{\mathbf{p}} I(X;Y) = \max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}) \\ &= \max_{\mathbf{p}} I(\mathbf{p};\frac{1}{2}\mathbf{Q}_1 + \frac{1}{2}\mathbf{Q}_2) \\ &\leq \frac{1}{2}\max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}_1) + \frac{1}{2}\max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}_2) \\ &= \frac{1}{2}C_1 + \frac{1}{2}C_2 \end{aligned}$$

Since $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are different, the inequality is strict. Hence :

$$C < \frac{1}{2}C_1 + \frac{1}{2}C_2$$

Problem 7.14 :

The capacity of a channel is :

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)] = \max_{p(x)}[H(X) - H(X|Y)]$$

Since in general $H(X|Y) \geq 0$ and $H(Y|X) \geq 0$, we obtain :

$$C \leq \min\{\max[H(Y)], \max[H(X)]\}$$

However, the maximum of $H(X)$ is attained when X is uniformly distributed, in which case $\max[H(X)] = \log|\mathcal{X}|$. Similarly : $\max[H(Y)] = \log|\mathcal{Y}|$ and by substituting in the previous inequality, we obtain

$$\begin{aligned} C &\leq \min\{\max[H(Y)], \max[H(X)]\} = \min\{\log|\mathcal{Y}|, \log|\mathcal{X}|\} \\ &= \min\{\log M, \log N\} \end{aligned}$$

Problem 7.15 :

(a) Let q be the probability of the input symbol 0, and therefore $(1-q)$ the probability of the input symbol 1. Then :

$$H(Y|X) = \sum_x P(x)H(Y|X=x)$$

$$
\begin{aligned}
&= qH(Y|X=0) + (1-q)H(Y|X=1) \\
&= (1-q)H(Y|X=1) = (1-q)H(\epsilon)
\end{aligned}
$$

The probability mass function of the output symbols is :

$$
\begin{aligned}
P(Y=0) &= qP(Y=0|X=0) + (1-q)P(Y=0|X=1) \\
&= q + (1-q)(1-\epsilon) = 1 - \epsilon + q\epsilon \\
P(Y=1) &= (1-q)\epsilon = \epsilon - q\epsilon
\end{aligned}
$$

Hence :

$$
C = \max_q [H(\epsilon - q\epsilon) - (1-q)H(\epsilon)]
$$

To find the probability q that achieves the maximum, we set the derivative of C with respect to q equal to 0. Thus :

$$
\frac{\vartheta C}{\vartheta q} = 0 = H(\epsilon) + \epsilon \log_2(\epsilon - q\epsilon) - \epsilon \log_2(1 - \epsilon + q\epsilon)
$$

Therefore :

$$
\log_2 \frac{\epsilon - q\epsilon}{1 - \epsilon + q\epsilon} = -\frac{H(\epsilon)}{\epsilon} \Longrightarrow q = \frac{\epsilon + 2^{-\frac{H(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})}
$$

and the channel capacity is

$$
C = H\left(\frac{2^{-\frac{H(\epsilon)}{\epsilon}}}{1 + 2^{-\frac{H(\epsilon)}{\epsilon}}}\right) - \frac{H(\epsilon) 2^{-\frac{H(\epsilon)}{\epsilon}}}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})}
$$

(b) If $\epsilon \to 0$, then using L'Hospital's rule we find that

$$
\lim_{\epsilon \to 0} \frac{H(\epsilon)}{\epsilon} = \infty, \qquad \lim_{\epsilon \to 0} \frac{H(\epsilon)}{\epsilon} 2^{-\frac{H(\epsilon)}{\epsilon}} = 0
$$

and therefore

$$
\lim_{\epsilon \to 0} C(\epsilon) = H(0) = 0
$$

If $\epsilon = 0.5$, then $H(\epsilon) = 1$ and $C = H(\frac{1}{5}) - \frac{2}{5} = 0.3219$. In this case the probability of the input symbol 0 is

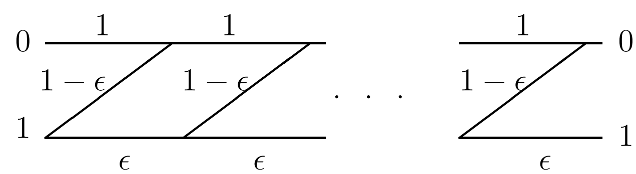
$$
q = \frac{\epsilon + 2^{-\frac{H(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{H(\epsilon)}{\epsilon}})} = \frac{0.5 + 0.25 \times (0.5 - 1)}{0.5 \times (1 + 0.25)} = \frac{3}{5}
$$

If $\epsilon = 1$, then $C = H(0.5) = 1$. The input distribution that achieves capacity is $P(0) = P(1) = 0.5$.

(c) The following figure shows the topology of the cascade channels. If we start at the input labeled 0, then the output will be 0. If however we transmit a 1, then the output will be zero with probability

$$\begin{aligned} P(Y=0|X=1) &= (1-\epsilon)+\epsilon(1-\epsilon)+\epsilon^2(1-\epsilon)+\cdots \\ &= (1-\epsilon)(1+\epsilon+\epsilon^2+\cdots) \\ &= 1-\epsilon\frac{1-\epsilon^n}{1-\epsilon} = 1-\epsilon^n \end{aligned}$$

Thus, the resulting system is equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.



(d) As $n \to \infty$, $\epsilon^n \to 0$ and the capacity of the channel goes to 0.

Problem 7.16 :

The SNR is :

$$\text{SNR} = \frac{2P}{N_0 2W} = \frac{P}{2W} = \frac{10}{10^{-9}\times 10^6} = 10^4$$

Thus the capacity of the channel is :

$$C = W\log_2(1+\frac{P}{N_0W}) = 10^6\log_2(1+10000) \approx 13.2879\times 10^6 \text{ bits/sec}$$

Problem 7.17 :

The capacity of the additive white Gaussian channel is :

$$C = \frac{1}{2}\log\left(1+\frac{P}{N_0W}\right)$$

For the nonwhite Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, the capacity is higher, The reason is that since noise samples are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore reduces their effective variance.

Problem 7.18 :

(a) The capacity of the binary symmetric channel with crossover probability ϵ is :

$$C = 1 - H(\epsilon)$$

where $H(\epsilon)$ is the binary entropy function. The rate distortion function of a zero mean Gaussian source with variance σ^2 per sample is :

$$R(D) = \begin{cases} \frac{1}{2}\log_2 \frac{\sigma^2}{D} & D \leq \sigma^2 \\ 0 & D > \sigma^2 \end{cases}$$

Since $C > 0$, we obtain :

$$\frac{1}{2}\log_2 \frac{\sigma^2}{D} \leq 1 - H(\epsilon) \Longrightarrow \frac{\sigma^2}{2^{2(1-H(\epsilon))}} \leq D$$

and therefore, the minimum value of the distortion attainable at the output of the channel is :

$$D_{\min} = \frac{\sigma^2}{2^{2(1-H(\epsilon))}}$$

(b) The capacity of the additive Gaussian channel is :

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma_n^2}\right)$$

Hence :

$$\frac{1}{2}\log_2 \frac{\sigma^2}{D} \leq C \Longrightarrow \frac{\sigma^2}{2^{2C}} \leq D \Longrightarrow \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}} \leq D$$

The minimum attainable distortion is :

$$D_{\min} = \frac{\sigma^2}{1 + \frac{P}{\sigma_n^2}}$$

(c) Here the source samples are dependent and therefore one sample provides information about the other samples. This means that we can achieve better results compared to the memoryless case at a given rate. In other words the distortion at a given rate for a source with memory is less than the distortion for a comparable source with memory. Differential coding methods discussed in Chapter 3 are suitable for such sources.

Problem 7.19 :

(a) The entropy of the source is :

$$H(X) = H(0.3) = 0.8813$$

and the capacity of the channel :

$$C = 1 - H(0.1) = 1 - 0.469 = 0.531$$

If the source is directly connected to the channel, then the probability of error at the destination is :

$$\begin{aligned} P(\text{error}) &= P(X=0)P(Y=1|X=0) + P(X=1)P(Y=0|X=1) \\ &= 0.3 \times 0.1 + 0.7 \times 0.1 = 0.1 \end{aligned}$$

(b) Since $H(X) > C$, some distortion at the output of the channel is inevitable. To find the minimum distortion, we set $R(D) = C$. For a Bernoulli type of source :

$$R(D) = \begin{cases} H(p) - H(D) & 0 \leq D \leq \min(p, 1-p) \\ 0 & \text{otherwise} \end{cases}$$

and therefore, $R(D) = H(p) - H(D) = H(0.3) - H(D)$. If we let $R(D) = C = 0.531$, we obtain

$$H(D) = 0.3503 \Longrightarrow D = \min(0.07, 0.93) = 0.07$$

The probability of error is :

$$P(\text{error}) \leq D = 0.07$$

(c) For reliable transmission we must have : $H(X) = C = 1 - H(\epsilon)$. Hence, with $H(X) = 0.8813$ we obtain

$$0.8813 = 1 - H(\epsilon) \Longrightarrow \epsilon < 0.016 \text{ or } \epsilon > 0.984$$

Problem 7.20 :

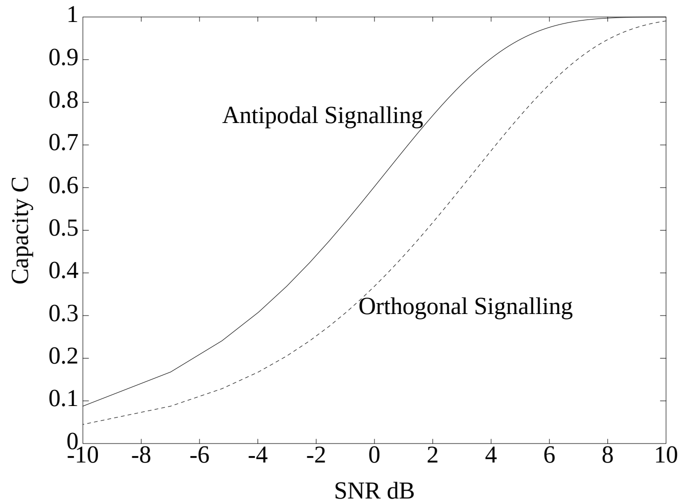
Both channels can be viewed as binary symmetric channels with crossover probability the probability of decoding a bit erroneously. Since :

$$P_b = \begin{cases} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] & \text{antipodal signalling} \\ Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] & \text{orthogonal signalling} \end{cases}$$

the capacity of the channel is :

$$C = \begin{cases} 1 - H\left(Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]\right) & \text{antipodal signalling} \\ 1 - H\left(Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]\right) & \text{orthogonal signalling} \end{cases}$$

In the next figure we plot the capacity of the channel as a function of $\frac{\mathcal{E}_b}{N_0}$ for the two signalling schemes.



Problem 7.21 :

(a) Since for each time slot $[mT, (m+1)T]$ we have $f_1(t) = \pm f_2(t)$, the signals are dependent and thus only one dimension is needed to represent them in the interval $[mT, (m+1)T]$. In this case the dimensionality of the signal space is upper bounded by the number of the different time slots used to transmit the message signals.

(b) If $f_1(t) \neq \alpha f_2(t)$, then the dimensionality of the signal space over each time slot is at most 2. Since there are n slots over which we transmit the message signals, the dimensionality of the signal space is upper bounded by $2n$.

(c) Let the decoding rule be that the first codeword is decoded when $\mathbf{r}$ is received if

$$p(\mathbf{r}|\mathbf{x}_1) > p(\mathbf{r}|\mathbf{x}_2)$$

The set of $\mathbf{r}$ that decode into $\mathbf{x}_1$ is

$$R_1 = \{\mathbf{r} : p(\mathbf{r}|\mathbf{x}_1) > p(\mathbf{r}|\mathbf{x}_2)\}$$

The characteristic function of this set $\chi_1(\mathbf{r})$ is by definition equal to 0 if $\mathbf{r} \notin R_1$ and equal to 1 if $\mathbf{r} \in R_1$. The characteristic function can be bounded as

$$1 - \chi_1(\mathbf{r}) \leq \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

This inequality is true if $\chi(\mathbf{r}) = 1$ because the right side is nonnegative. It is also true if $\chi(\mathbf{r}) = 0$ because in this case $p(\mathbf{r}|\mathbf{x}_2) > p(\mathbf{r}|\mathbf{x}_1)$ and therefore,

$$1 \leq \frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)} \Longrightarrow 1 \leq \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

Given that the first codeword is sent, then the probability of error is

$$\begin{aligned} P(\text{error}|\mathbf{x}_1) &= \int \cdots \int_{R^N - R_1} p(\mathbf{r}|\mathbf{x}_1) d\mathbf{r} \\ &= \int \cdots \int_{R^N} p(\mathbf{r}|\mathbf{x}_1)[1 - \chi_1(\mathbf{r})] d\mathbf{r} \\ &\leq \int \cdots \int_{R^N} p(\mathbf{r}|\mathbf{x}_1) \left(\frac{p(\mathbf{r}|\mathbf{x}_2)}{p(\mathbf{r}|\mathbf{x}_1)}\right)^{\frac{1}{2}} d\mathbf{r} \\ &= \int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_1) p(\mathbf{r}|\mathbf{x}_2)} d\mathbf{r} \end{aligned}$$

(d) The result follows immediately if we use the union bound on the probability of error. Thus, assuming that $\mathbf{x}_m$ was transmitted, then taking the signals $\mathbf{x}_{m'}$, $m' \neq m$, one at a time and ignoring the presence of the rest, we can write

$$P(\text{error}|\mathbf{x}_m) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} \int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_m) p(\mathbf{r}|\mathbf{x}_{m'})} d\mathbf{r}$$

(e) Let $\mathbf{r} = \mathbf{x}_m + \mathbf{n}$ with $\mathbf{n}$ an N-dimensional zero-mean Gaussian random variable with variance per dimension equal to $\sigma^2 = \frac{N_0}{2}$. Then,

$$p(\mathbf{r}|\mathbf{x}_m) = p(\mathbf{n}) \qquad \text{and} \qquad p(\mathbf{r}|\mathbf{x}_{m'}) = p(\mathbf{n} + \mathbf{x}_m - \mathbf{x}_{m'})$$

and therefore :

$$\begin{aligned} &\int \cdots \int_{R^N} \sqrt{p(\mathbf{r}|\mathbf{x}_m) p(\mathbf{r}|\mathbf{x}_{m'})} d\mathbf{r} \\ &= \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{4}}} e^{-\frac{|\mathbf{n}|^2}{2N_0}} \frac{1}{(\pi N_0)^{\frac{N}{4}}} e^{-\frac{|\mathbf{n}+\mathbf{x}_m-\mathbf{x}_{m'}|^2}{2N_0}} d\mathbf{n} \\ &= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}} \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{2|\mathbf{n}|^2+|\mathbf{x}_m-\mathbf{x}_{m'}|^2/2+2\mathbf{n}\cdot(\mathbf{x}_m-\mathbf{x}_{m'})}{2N_0}} d\mathbf{n} \\ &= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}} \int \cdots \int_{R^N} \frac{1}{(\pi N_0)^{\frac{N}{2}}} e^{-\frac{|\mathbf{n}+\frac{\mathbf{x}_m-\mathbf{x}_{m'}}{2}|^2}{N_0}} d\mathbf{n} \\ &= e^{-\frac{|\mathbf{x}_m-\mathbf{x}_{m'}|^2}{4N_0}} \end{aligned}$$

Using the union bound in part (d), we obtain :

$$P(\text{error}|x_m(t) \text{ sent}) \leq \sum_{\substack{1 \leq m' \leq M \\ m' \neq m}} e^{-\frac{|\mathbf{x}_m - \mathbf{x}_{m'}|^2}{4N_0}}$$

Problem 7.22 :

Equation (7.3-2) gives that the cutoff rate R_2 with two quantization levels is

$$R_2 = \max_{p_{in}} \left\{ -\log_2 \sum_{out=0}^{1} \left[\sum_{in=0}^{1} p_{in} \sqrt{P(out|in)} \right]^2 \right\}$$

By naming the argument of the $\log_2$ function as S, the above corresponds to

$$R_2 = -\log_2 \min_{p_{in}} S$$

Suppose the probabilities of the input symbols are $p_0 = x,\ p_1 = 1 - x$. Also, the probability of error for the BSC is p, where p is the error rate for the modulation method employed. Then

$$\begin{aligned} S &= [x\sqrt{1-p} + (1-x)\sqrt{x}]^2 + [x\sqrt{p} + (1-x)\sqrt{1-p}]^2 \\ &= 2x^2(1 - 2\sqrt{p(1-p)}) - 2x(1 - 2\sqrt{p(1-p)}) + 1 \end{aligned}$$

By taking the first derivative of S w.r.t. x we find that the extremum point is

$$\frac{dS}{dx} = 0 \Rightarrow 4x(1 - 2\sqrt{p(1-p)}) - 2(1 - 2\sqrt{p(1-p)}) = 0 \Rightarrow x = 1/2$$

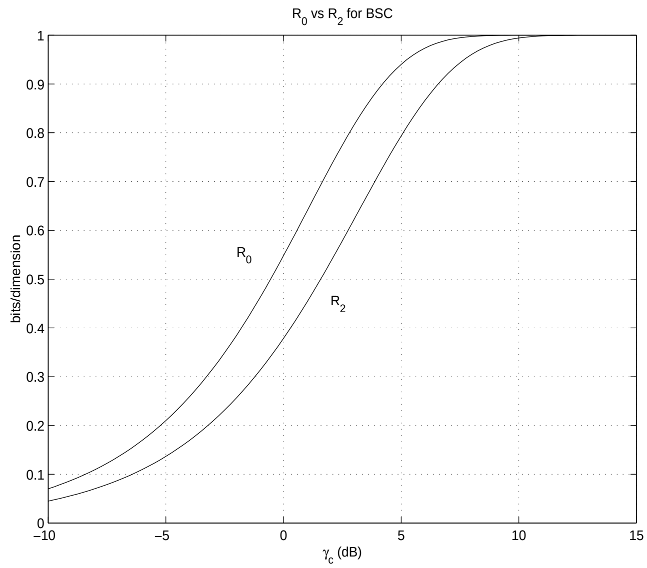
and the corresponding S is

$$\min S = S_{|x=1/2} = \frac{1 + 2\sqrt{p(1-p)}}{2}$$

Hence,

$$R_2 = -\log_2 \min S = 1 - \log_2 \left[1 + \sqrt{4p(1-p)}\right]$$

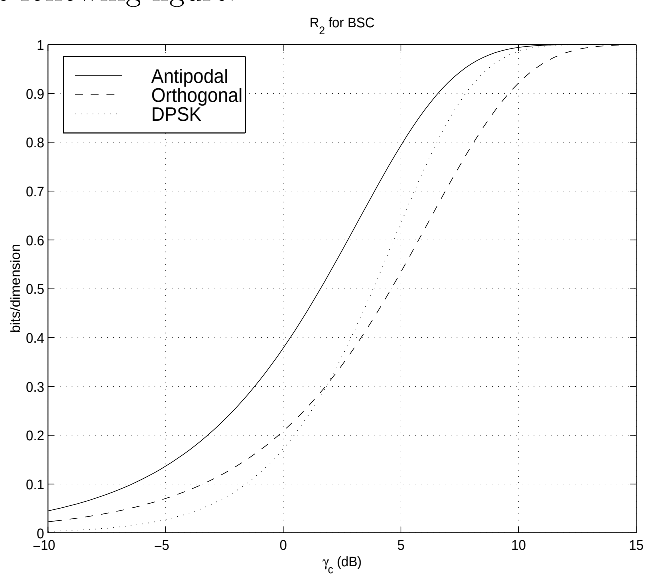
The plot of the comparison between R_0 and R_2 is given in the following figure:



As we see, the loss in performance when a two-level hard decision (instead of a soft-decision) is employed is approximately 2 dB.

Problem 7.23 :

The plot with the cutoff rate R_2 for the BSC, when the three different modulation schemes are employed is given in the following figure:



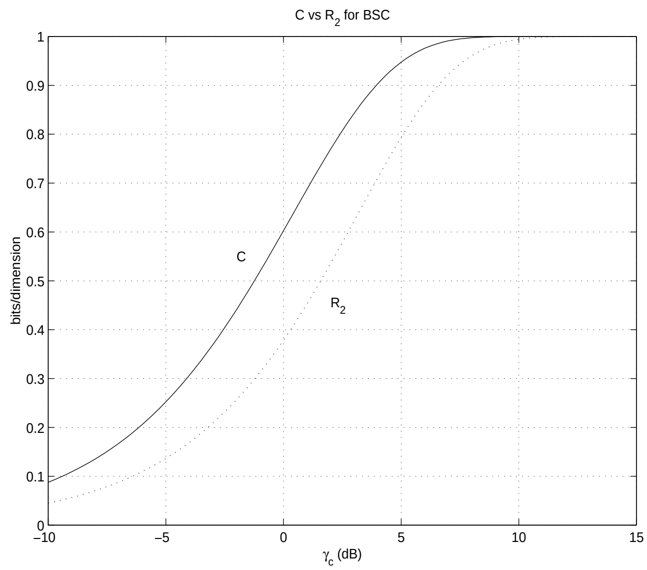
As we see, orthogonal signaling is 3 dB worse than antipodal signaling. Also, DPSK is the worst scheme in very low SNR's, but approaches antipodal signaling performance as the SNR goes up. Both these conclusions agree with the well-known results on the performance of these schemes, as given by their error probabilities given in Chapter 5.

Problem 7.24 :

Remember that the capacity of the BSC is

$$C = p\log_2 2p + (1-p)\log_2(2(1-p))$$

where p is the error probability (for binary antipodal modulation for this particular case). Then, the plot with the comparison of the capacity vs the cutoff rate R_2 for the BSC, with antipodal signaling, is given in the following figure:



We notice that the difference between the hard-decision cutoff rate and the capacity of the channel is approximately 2.5 to 3 dB.

Problem 7.25 :

From expression (7.2-31) we have that

$$R_0 = -\log_2\left(\sum_{l=1}^{M}\sum_{m=1}^{M} p_l p_m e^{-d_{lm}^2/4N_0}\right)$$

and, since we are given that equiprobable input symbols maximize R_0, $p_l = p_m = 1/M$ and the above expression becomes

$$R_0 = -\log_2\left(\frac{1}{M^2}\sum_{l=1}^{M}\sum_{m=1}^{M} e^{-d_{lm}^2/4N_0}\right)$$

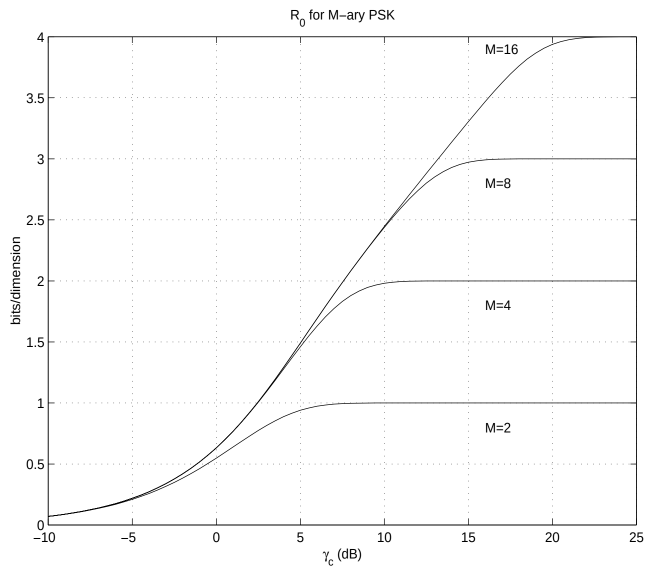
The M-ary PSK constellation points are symmetrically spaced around the unit circle. Hence, the sum of the distances between them is the same, independent of the reference point, or $\sum_{m=1}^{M} e^{-d_{lm}^2/4N_0}$ is the same for any $l = 0, 1, ...M-1$. Hence,

$$
\begin{aligned}
R_0 &= -\log_2\left(\frac{M}{M^2}\sum_{l=1}^{M} e^{-d_{0m}^2/4N_0}\right) \\
&= \log_2 M - \log_2 \sum_{l=1}^{M} e^{-d_{0m}^2/4N_0}
\end{aligned}
$$

The distance of equally spaced points around a circle with radius $\sqrt{\mathcal{E}_c}$ is $d_m = 2\sqrt{\mathcal{E}_c}\sin\frac{m\pi}{M}$. So

$$
R_0 = \log_2 M - \log_2 \sum_{l=1}^{M} e^{-(\mathcal{E}_c/N_0)\sin^2\frac{m\pi}{M}}
$$

The plot of R_0 for the various levels of M-ary PSK is given in the following figure:



CHAPTER 8

Problem 8.1 :

(a) Interchanging the first and third rows, we obtain the systematic form :

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(b)

$$\mathbf{H} = \left[\mathbf{P}^T | \mathbf{I}_4\right] = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Since we have a (7,3) code, there are $2^3 = 8$ valid codewords, and 2^4 possible syndromes. From these syndromes the all-zero one corresponds to no error, 7 will correspond to single errors and 8 will correspond to double errors (the choice is not unique) :

Error pattern	Syndrome
0 0 0 0 0 0 0	0 0 0 0
0 0 0 0 0 0 1	0 0 0 1
0 0 0 0 0 1 0	0 0 1 0
0 0 0 0 1 0 0	0 1 0 0
0 0 0 1 0 0 0	1 0 0 0
0 0 1 0 0 0 0	1 1 0 1
0 1 0 0 0 0 0	0 1 1 1
1 0 0 0 0 0 0	1 1 1 0
1 0 0 0 0 0 1	1 1 1 1
1 0 0 0 0 1 0	1 1 0 0
1 0 0 0 1 0 0	1 0 1 0
1 0 0 1 0 0 0	0 1 1 0
1 0 1 0 0 0 0	0 0 1 1
1 1 0 0 0 0 0	1 0 0 1
0 1 0 0 0 1 0	0 1 0 1
0 0 0 1 1 0 1	1 0 1 1

(d) We note that there are 3 linearly independent columns in $\mathbf{H}$, hence there is a codeword $\mathbf{C}_m$ with weight w_m =4 such that $\mathbf{C}_m \mathbf{H}^T = 0$. Accordingly : $d_{\min} = 4$. This can be also obtained by generating all 8 codewords for this code and checking their minimum weight.

(e) 101 generates the codeword : $101 \rightarrow \mathbf{C} = 1010011$. Then : $\mathbf{C}\mathbf{H}^T = [0000]$.

Problem 8.2 :

$$\mathbf{G}_a = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{G}_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Message $\mathbf{X}_{\rm m}$	$\mathbf{C}_{\rm ma} = \mathbf{X}_{\rm m}\mathbf{G}_{\rm a}$	$\mathbf{C}_{\rm mb} = \mathbf{X}_{\rm m}\mathbf{G}_{\rm b}$
0 0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 1 1	0 0 0 1 0 1 1
0 0 1 0	0 0 1 0 1 1 0	0 0 1 0 1 1 0
0 0 1 1	0 0 1 1 1 0 1	0 0 1 1 1 0 1
0 1 0 0	0 1 0 1 1 0 0	0 1 0 0 1 1 1
0 1 0 1	0 1 0 0 1 1 1	0 1 0 1 1 0 0
0 1 1 0	0 1 1 1 0 1 0	0 1 1 0 0 0 1
0 1 1 1	0 1 1 0 0 0 1	0 1 1 1 0 1 0
1 0 0 0	1 0 1 1 0 0 0	1 0 0 0 1 0 1
1 0 0 1	1 0 1 0 0 1 1	1 0 0 1 1 1 0
1 0 1 0	1 0 0 1 1 1 0	1 0 1 0 0 1 1
1 0 1 1	1 0 0 0 1 0 1	1 0 1 1 0 0 0
1 1 0 0	1 1 1 0 1 0 0	1 1 0 0 0 1 0
1 1 0 1	1 1 1 1 1 1 1	1 1 0 1 0 0 1
1 1 1 0	1 1 0 0 0 1 0	1 1 1 0 1 0 0
1 1 1 1	1 1 0 1 0 0 1	1 1 1 1 1 1 1

As we see, the two generator matrices generate the same set of codewords.

Problem 8.3 :

The weight distribution of the (7,4) Hamming code is $(n = 7)$:

$$\begin{aligned} A(x) &= \tfrac{1}{8}\left[(1+x)^7 + 7(1+x)^3(1-x)^4\right] \\ &= \tfrac{1}{8}\left[8 + 56x^3 + 56x^4 + 8x^7\right] \\ &= 1 + 7x^3 + 7x^4 + x^7 \end{aligned}$$

Hence, we have 1 codeword of weight zero, 7 codewords of weight 3, 7 codewords of weight 4, and one codeword of weight 7. which agrees with the codewords given in Table 8-1-2.

Problem 8.4:

(a) The generator polynomial for the (15,11) Hamming code is given as $g(p) = p^4 + p + 1$. We will express the powers p^l as : $p^l = Q_l(p)g(p) + R_l(p)$ $l = 4, 5, ...14$, and the polynomial $R_l(p)$ will give the parity matrix $\mathbf{P}$, so that $\mathbf{G}$ will be $\mathbf{G} = [\mathbf{I}_{11}|\mathbf{P}]$. We have :

$$\begin{array}{lll} p^4 & = & g(p) + p + 1 \\ p^5 & = & pg(p) + p^2 + p \\ p^6 & = & p^2 g(p) + p^3 + p^2 \\ p^7 & = & (p^3+1)g(p) + p^3 + p + 1 \\ p^8 & = & (p^4+p+1)g(p) + p^2 + 1 \\ p^9 & = & (p^5+p^2+p)g(p) + p^3 + p \\ p^{10} & = & (p^6+p^3+p^2+1)g(p) + p^2 + p + 1 \\ p^{11} & = & (p^7+p^4+p^3+p)g(p) + p^3 + p^2 + p \\ p^{12} & = & (p^8+p^5+p^4+p^2+1)g(p) + p^3 + p^2 + p + 1 \\ p^{13} & = & (p^9+p^6+p^5+p^3+p+1)g(p) + p^3 + p^2 + 1 \\ p^{14} & = & (p^{10}+p^7+p^6+p^4+p^2+p+1)g(p) + p^3 + 1 \end{array}$$

Using $R_l(p)$ (with $l = 4$ corresponding to the last row of $\mathbf{G}$,... $l = 14$ corresponding to the first row) for the parity matrix $\mathbf{P}$ we obtain :

$$\mathbf{G} = \left[\begin{array}{ccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&1&1&0&1\\
0&0&1&0&0&0&0&0&0&0&0&1&1&1&1\\
0&0&0&1&0&0&0&0&0&0&0&1&1&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&1&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&1
\end{array}\right]$$

(b) In order to obtain the generator polynomial for the dual code, we first factor $p^{15} + 1$ into : $p^{15+1} = g(p)h(p)$ to obtain the parity polynomial $h(p) = (p^{15} + 1)/g(p) = p^{11} + p^8 + p^7 + p^5 + p^3 + p^2 + p + 1$. Then, the generator polynomial for the dual code is given by :

$$p^{11} h(p^{-1}) = 1 + p^3 + p^4 + p^6 + p^8 + p^9 + p^{10} + p^{11}$$

Problem 8.5 :

We can determine $\mathbf{G}$, in a systematic form, from the generator polynomial $g(p) = p^3 + p^2 + 1$:

$$\begin{array}{l} p^6 = (p^3+p^2+p)g(p) + p^2 + p \\ p^5 = (p^2+p+1)g(p) + p + 1 \\ p^4 = (p+1)g(p) + p^2 + p + 1 \\ p^3 = g(p) + p^2 + 1 \end{array} \qquad \mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Hence, the parity check matrix for the extended code will be (according to 8-1-15) :

$$\mathbf{H}_e = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and in systematic form (we add rows 1,2,3 to the last one) :

$$\mathbf{H}_{es} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{G}_{es} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Note that $\mathbf{G}_{es}$ can be obtained from the generator matrix $\mathbf{G}$ for the initial code, by adding an overall parity check bit. The code words for the extended systematic code are :

Message $\mathbf{X}_{\mathrm{m}}$	Codeword $\mathbf{C}_{\mathrm{m}}$
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 1 0 1 1
0 0 1 0	0 0 1 0 1 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 0 1 1 1
0 1 0 1	0 1 0 1 1 1 0 0
0 1 1 0	0 1 1 0 1 0 0 1
0 1 1 1	0 1 1 1 0 0 1 0
1 0 0 0	1 0 0 0 1 1 0 1
1 0 0 1	1 0 0 1 0 1 1 0
1 0 1 0	1 0 1 0 0 0 1 1
1 0 1 1	1 0 1 1 1 0 0 0
1 1 0 0	1 1 0 0 1 0 1 0
1 1 0 1	1 1 0 1 0 0 0 1
1 1 1 0	1 1 1 0 0 1 0 0
1 1 1 1	1 1 1 1 1 1 1 1

An alternative way to obtain the codewords for the extended code is to add an additional check bit to the codewords of the initial (7,4) code which are given in Table 8-1-2. As we see, the minimum weight is 4 and hence : $d_{\min} = 4$.

Problem 8.6 :

(a) We have obtained the generator matrix **G** for the (15,11) Hamming code in the solution of Problem 8.4. The shortened code will have a generator matrix $\mathbf{G}_s$ obtained by **G**, by dropping its first 7 rows and the first 7 columns or :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Then the possible messages and the codewords corresponding to them will be :

Message $\mathbf{X}_m$	Codeword $\mathbf{C}_m$
0 0 0 0	0 0 0 0 0 0 0 0
0 0 0 1	0 0 0 1 0 0 1 1
0 0 1 0	0 0 1 0 0 1 1 0
0 0 1 1	0 0 1 1 0 1 0 1
0 1 0 0	0 1 0 0 1 1 0 0
0 1 0 1	0 1 0 1 1 1 1 1
0 1 1 0	0 1 1 0 1 0 1 0
0 1 1 1	0 1 1 1 1 0 0 1
1 0 0 0	1 0 0 0 1 0 1 1
1 0 0 1	1 0 0 1 1 0 0 0
1 0 1 0	1 0 1 0 1 1 0 1
1 0 1 1	1 0 1 1 1 0 1 0
1 1 0 0	1 1 0 0 0 1 1 1
1 1 0 1	1 1 0 1 0 1 0 0
1 1 1 0	1 1 1 0 0 0 0 1
1 1 1 1	1 1 1 1 0 0 1 0

(b) As we see the minimum weight and hence the minimum distance is 3 : $d_{\min} = 3$.

Problem 8.7 :

(a)

$$g(p) = (p^4 + p^3 + p^2 + p + 1)(p^4 + p + 1)(p^2 + p + 1) = p^{10} + p^8 + p^5 + p^4 + p^2 + p + 1$$

Factoring p^l, $l = 14, ...10$, into $p^l = g(p)Q_l(p) + R_l(p)$ we obtain the generator matrix in systematic form :

$$\left.\begin{array}{l} p^{14} = (p^4+p^2+1)g(p) + p^9+p^7+p^4+p^3+p+1 \\ p^{13} = (p^3+p)g(p) + p^9+p^8+p^7+p^6+p^4+p^2+p \\ p^{12} = (p^2+1)g(p) + p^8+p^7+p^6+p^5+p^3+p+1 \\ p^{11} = pg(p) + p^9+p^6+p^5+p^3+p^2+p \\ p^{10} = g(p) + p^8+p^5+p^4+p^2+p+1 \end{array}\right\} \Rightarrow$$

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&1&0&1&0&0&1&1&0&1&1 \\ 0&1&0&0&0&1&1&1&1&0&1&0&1&1&0 \\ 0&0&1&0&0&0&1&1&1&1&0&1&0&1&1 \\ 0&0&0&1&0&1&0&0&1&1&0&1&1&1&0 \\ 0&0&0&0&1&0&1&0&0&1&1&0&1&1&1 \end{bmatrix}$$

The codewords are obtained from the equation : $\mathbf{C}_m = \mathbf{X}_m\mathbf{G}$, where X_m is the row vector containing the five message bits.

(b)

$$d_{\min} = 7$$

(c) The error-correcting capability of the code is :

$$t = \left\lfloor \frac{d_{\min}-1}{2} \right\rfloor = 3$$

(d) The error-detecting capability of the code is : $d_{\min} - 1 = 6$.

(e)

$$g(p) = (p^{15}+1)/(p^2+p+1) = p^{13}+p^{12}+p^{10}+p^9+p^7+p^6+p^4+p^3+p+1$$

Then :

$$\begin{array}{l} p^{14} = (p+1)g(p) + p^{12}+p^{11}+p^9+p^8+p^6+p^5+p^3+p^2+1 \\ p^{13} = g(p) + p^{12}+p^{10}+p^9+p^7+p^6+p^4+p^3+p+1 \end{array}$$

Hence, the generator matrix is :

$$\mathbf{G} = \begin{bmatrix} 1&0&1&1&0&1&1&0&1&1&0&1&1&0&1 \\ 0&1&1&0&1&1&0&1&1&0&1&1&0&1&1 \end{bmatrix}$$

and the valid codewords :

$\mathbf{X}_{\mathrm{m}}$	Codeword $\mathbf{C}_{\mathrm{m}}$
0 0	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 1	0 1 1 0 1 1 0 1 1 0 1 1 0 1 1
1 0	1 0 1 1 0 1 1 0 1 1 0 1 1 0 1
1 1	1 1 0 1 1 0 1 1 0 1 1 0 1 1 0

The minimum distance is : $d_{\min} = 10$

Problem 8.8 :

The polynomial p^7+1 is factors as follows : $p^7+1 = (p+1)(p^3+p^2+1)(p^3+p+1)$. The generator polynomials for the matrices $\mathbf{G}_1, \mathbf{G}_2$ are : $g_1(p) = p^3+p^2+1,\ g_2(p) = p^3+p+1$. Hence the parity polynomials are : $h_1(p) = (p^7+1)/g_1(p) = p^4+p^3+p^2+1,\ h_2(p) = (p^7+1)/g_2(p) = p^4+p^2+p+1$. The generator polynomials for the matrices $\mathbf{H}_1, \mathbf{H}_2$ are : $p^4 h_1(p^{-1}) = 1+p+p^2+p^4,\ p^4 h_2(p^{-1}) = 1+p^2+p^3+p^4$. The rows of the matrices $\mathbf{H}_1, \mathbf{H}_2$ are given by : $p^i p^4 h_{1/2}(p^{-1}),\ i = 0, 1, 2$, so :

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Problem 8.9 :

We have already generated an extended (8,4) code from the (7,4) Hamming code in Probl. 8.5. Since the generator matrix for the (7,4) Hamming code is not unique, in this problem we will construct the extended code, starting from the generator matrix given in 8-1-7 :

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then :

$$H_e = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We can bring this parity matrix into systematic form by adding rows 1,2,3 into the fourth row :

$$H_{es} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then :

$$G_{e,s} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 8.10 :

$$G=\left[\begin{array}{cccccc}1&0&0&1&1&0\\0&1&0&0&1&1\\0&0&1&1&0&1\end{array}\right]\Rightarrow H=\left[\begin{array}{cccccc}1&0&1&1&0&0\\1&1&0&0&1&0\\0&1&1&0&0&1\end{array}\right]$$

Then the standard array is :

000	001	010	011	100	101	110	111
000000	001101	010011	011110	100110	101011	110101	111000
000001	001100	010010	011111	100111	101010	110100	111001
000010	001111	010001	011100	100100	101001	110111	111010
000100	001001	010111	011010	100010	101111	110001	111100
001000	000101	011011	010110	101110	100011	111101	110000
010000	011101	000011	001110	110110	111011	100101	101000
100000	101101	110011	111110	000110	001011	010101	011000
100001	101100	110010	111111	000111	001010	010100	011001

For each column, the first row is the message, the second row is the correct codeword corresponding to this message, and the rest of the rows correspond to the received words which are the sum of the valid codeword plus the corresponding error pattern (coset leader). The error patterns that this code can correct are given in the first column (all-zero codeword), and the corresponding syndromes are :

E_i	$S_i = E_i H^T$
000000	000
000001	001
000010	010
000100	100
001000	101
010000	011
100000	110
100001	111

We note that this code can correct all single errors and one two-bit error pattern.

Problem 8.11 :

$$G=\left[\begin{array}{ccccccc}1&0&0&1&0&1&1\\0&1&0&1&1&1&0\\0&0&1&0&1&1&1\end{array}\right]\Rightarrow H=\left[\begin{array}{ccccccc}1&1&0&1&0&0&0\\0&1&1&0&1&0&0\\1&1&1&0&0&1&0\\1&0&1&0&0&0&1\end{array}\right]$$

Then, the standard array will be :

000	001	010	011	100	101	110	111
0000000	0010111	0101110	0111001	1001011	1011100	1100101	1110010
0000001	0010110	0101111	0111000	1001010	1011101	1100100	1110011
0000010	0010101	0101101	0111011	1001001	1011110	1100111	1110000
0000100	0010011	0101010	0111101	1001111	1011000	1100001	1110110
0001000	0011111	0100110	0110001	1000011	1010100	1101101	1111010
0010000	0000111	0111110	0101001	1011011	1001100	1110101	1100010
0100000	0110111	0001110	0011001	1101011	1111100	1000101	1010010
1000000	1010111	1101110	1111001	0001011	0011100	0100101	0110010
1100000	1110111	1001110	1011001	0101011	0111100	0000101	0010010
1010000	1000111	1111110	1101001	0011011	0001100	0110101	0100010
1001000	1011111	1100110	1110001	0000011	0010100	0101101	0111010
1000100	1010011	1101010	1111101	0001111	0011000	0100001	0110110
1000010	1010101	1101100	1111010	0001001	0011110	0100111	0110000
1000001	1010110	1101111	1111001	0001010	0011101	0100100	0110011
0010001	0000110	0111111	0101001	1011010	1001101	1110100	1100011
0001101	0011010	0100011	0110101	1000110	1010001	1101000	1111111

For each column, the first row is the message, the second row is the correct codeword corresponding to this message, and the rest of the rows correspond to the received words which are the sum of the valid codeword plus the corresponding error pattern (coset leader). The error patterns that this code can correct are given in the first column (all-zero codeword), and the corresponding syndromes are :

E_i	$S_i = E_i H^T$
0000000	0000
0000001	0001
0000010	0010
0000100	0100
0001000	1000
0010000	0111
0100000	1110
1000000	1011
1100000	0101
1010000	11000
1001000	0011
1000100	1111
1000010	1001
1000001	1010
0010001	0110
0001101	1101

We note that this code can correct all single errors, seven two-bit error patterns, and one three-bit error pattern.

Problem 5.12 :

The generator matrix for the systematic (7,4) cyclic Hamming code is given by (8-1-37) as :

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Then, the correctable error patterns E_i with the corresponding syndrome $S_i = E_i H^T$ are :

S_i	E_i
000	0000000
001	0000001
010	0000010
011	0001000
100	0000100
101	1000000
110	0100000
111	0010000

Problem 8.13 :

We know that : $\mathbf{e}_1 + \mathbf{e}_2 = \mathbf{C}$, where $\mathbf{C}$ is a valid codeword. Then :

$$\mathbf{S}_1 + \mathbf{S}_2 = \mathbf{e}_1 \mathbf{H}^T + \mathbf{e}_2 \mathbf{H}^T = (\mathbf{e}_1 + \mathbf{e}_2)\mathbf{H}^T = \mathbf{C}\mathbf{H}^T = \mathbf{0}$$

since a valid codeword is orthogonal to the parity matrix. Hence : $\mathbf{S}_1 + \mathbf{S}_2 = \mathbf{0}$, and since modulo-2 addition is the same with modulo-2 subtraction :

$$\mathbf{S}_1 - \mathbf{S}_2 = \mathbf{0} \Rightarrow \mathbf{S}_1 = \mathbf{S}_2$$

Problem 8.14 :

(a) Let $g(p) = p^8 + p^6 + p^4 + p^2 + 1$ be the generator polynomial of an (n, k) cyclic code. Then, $n - k = 8$ and the rate of the code is

$$R = \frac{k}{n} = 1 - \frac{8}{n}$$

The rate R is minimum when $\frac{8}{n}$ is maximum subject to the constraint that R is positive. Thus, the first choice of n is $n = 9$. However, the generator polynomial $g(p)$ does not divide $p^9 + 1$ and therefore, it can not generate a $(9, 1)$ cyclic code. The next candidate value of n is 10. In this case

$$p^{10} + 1 = g(p)(p^2 + 1)$$

and therefore, $n = 10$ is a valid choice. The rate of the code is $R = \frac{k}{n} = \frac{2}{10} = \frac{1}{5}$.

(b) In the next table we list the four codewords of the $(10, 2)$ cyclic code generated by $g(p)$.

Input	$X(p)$	Codeword
00	0	0000000000
01	1	0101010101
10	p	1010101010
11	$p+1$	1111111111

As it is observed from the table, the minimum weight of the code is 5 and since the code is linear $d_{\min} = w_{\min} = 5$.

(c) The coding gain of the $(10, 2)$ cyclic code in part (a) is

$$G_{\text{coding}} = d_{\min} R = 5 \times \frac{2}{10} = 1$$

Problem 8.15 :

(a) For every n

$$p^n + 1 = (p + 1)(p^{n-1} + p^{n-2} + \cdots + p + 1)$$

where additions are modulo 2. Since $p + 1$ divides $p^n + 1$ it can generate a (n, k) cyclic code, where $k = n - 1$.

(b) The i^{th} row of the generator matrix has the form

$$\mathbf{g}_i = [\ 0 \quad \cdots \quad 0 \quad 1 \quad 0 \quad \cdots \quad 0 \quad p_{i,1}\ ]$$

where the 1 corresponds to the i-th column (to give a systematic code) and the $p_{i,1}$, $i = 1, \ldots, n-1$, can be found by solving the equations

$$p^{n-i} + p_{i,1} = p^{n-i} \bmod p + 1, \qquad 1 \leq i \leq n - 1$$

Since $p^{n-i} \bmod p + 1 = 1$ for every i, the generator and the parity check matrix are given by

$$\mathbf{G} = \begin{pmatrix} 1 & \cdots & 0 & | & 1 \\ \vdots & \ddots & \vdots & | & \vdots \\ 0 & \cdots & 1 & | & 1 \end{pmatrix}, \qquad \mathbf{H} = [\ 1 \quad 1 \quad \cdots \quad 1 \quad | \quad 1\ ]$$

(c) A vector $\mathbf{c} = [c_1, c_2, \ldots, c_n]$ is a codeword of the $(n, n-1)$ cyclic code if it satisfies the condition $\mathbf{cH}^t = 0$. But,

$$\mathbf{cH}^t = 0 = \mathbf{c} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 + c_2 + \cdots c_n$$

Thus, the vector $\mathbf{c}$ belongs to the code if it has an even weight. Therefore, the cyclic code generated by the polynomial $p+1$ is a simple parity check code.

Problem 8.16 :

(a) The generator polynomial of degree $4 = n - k$ should divide the polynomial $p^6 + 1$. Since the polynomial $p^6 + 1$ assumes the factorization

$$p^6 + 1 = (p+1)^3(p+1)^3 = (p+1)(p+1)(p^2+p+1)(p^2+p+1)$$

we find that the shortest possible generator polynomial of degree 4 is

$$g(p) = p^4 + p^2 + 1$$

The i^{th} row of the generator matrix $\mathbf{G}$ has the form

$$\mathbf{g}_i = \left[\begin{array}{cccccccccc} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & p_{i,1} & \cdots & p_{i,4} \end{array} \right]$$

where the 1 corresponds to the i-th column (to give a systematic code) and the $p_{i,1}, \ldots, p_{i,4}$ are obtained from the relation

$$p^{6-i} + p_{i,1}p^3 + p_{i,2}p^2 p_{i,3}p + p_{i,4} = p^{6-i}(\bmod\ p^4 + p^2 + 1)$$

Hence,

$$\begin{aligned} p^5 \bmod p^4 + p^2 + 1 &= (p^2+1)p \bmod p^4 + p^2 + 1 = p^3 + p \\ p^4 \bmod p^4 + p^2 + 1 &= p^2 + 1 \bmod p^4 + p^2 + 1 = p^2 + 1 \end{aligned}$$

and therefore,

$$\mathbf{G} = \left(\begin{array}{cc|cccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right)$$

The codewords of the code are

$$\begin{aligned} \mathbf{c}_1 &= [\ 0\ 0\ 0\ 0\ 0\ 0\ ] \\ \mathbf{c}_2 &= [\ 1\ 0\ 1\ 0\ 1\ 0\ ] \\ \mathbf{c}_3 &= [\ 0\ 1\ 0\ 1\ 0\ 1\ ] \\ \mathbf{c}_4 &= [\ 1\ 1\ 1\ 1\ 1\ 1\ ] \end{aligned}$$

(b) The minimum distance of the linear $(6,2)$ cyclic code is $d_{\min} = w_{\min} = 3$. Therefore, the code can correct

$$e_c = \frac{d_{\min} - 1}{2} = 1 \text{ error}$$

Problem 8.17 :

Consider two n-tuples in the same row of a standard array. Clearly, if $\mathbf{Y}_1, \mathbf{Y}_2$ denote the n-tuples, $\mathbf{Y}_1 = \mathbf{C}_j + \mathbf{e}$, $\mathbf{Y}_2 = \mathbf{C}_k + \mathbf{e}$, where $\mathbf{C}_k, \mathbf{C}_j$ are two valid codewords, and the error pattern e is the same since they are in the same row of the standard array. Then :

$$\mathbf{Y}_1 + \mathbf{Y}_2 = \mathbf{C}_j + \mathbf{e} + \mathbf{C}_k + \mathbf{e} = \mathbf{C}_j + \mathbf{C}_k = \mathbf{C}_m$$

where $\mathbf{C}_m$ is another valid codeword (this follows from the linearity of the code).

Problem 8.18 :

From Table 8-1-6 we find that the coefficients of the generator polynomial for the (15,7) BCH code are $721 \rightarrow 111010001$ or $g(p) = p^8 + p^7 + p^6 + p^4 + 1$. Then, we can determine the l-th row of the generator matrix $\mathbf{G}$, using the modulo $R_l(p) \ : p^{n-l} = Q_l(p)g(p) + R_l(p), \ l = 1, 2, ..., 7$. Since the generator matrix of the shortened code is obtained by removing the first three rows of $\mathbf{G}$, we perform the above calculations for $l = 4, 5, 6, 7$, only :

$$\begin{array}{l} p^{11} = (p^3 + p^2 + 1)g(p) + p^4 + p^3 + p^2 + 1 \\ p^{10} = (p^2 + p)g(p) + p^7 + p^6 + p^5 + p^2 + p \\ p^9 = (p+1)g(p) + p^6 + p^5 + p^4 + p + 1 \\ p^8 = (p+1)g(p) + p^7 + p^6 + p^4 + 1 \end{array}$$

Hence :

$$\mathbf{G}_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 8.19 :

For M-ary FSK detected coherently, the bandwidth expansion factor is :

$$\left(\frac{W}{R}\right)_{FSK} = \frac{M}{2\log_2 M}$$

For the Hadamard code : In time T (block transmission time), we want to transmit n bits, so for each bit we have time : $T_b = T/n$. Since for each bit we use binary PSK, the bandwidth requirement is approximately : $W = 1/T_b = n/T$. But $T = k/R$, hence :

$$W = \frac{n}{k}R \Rightarrow \frac{W}{R} = \frac{n}{k}$$

(this is a general result for binary block-encoded signals). For the specific case of a Hadamard code the number of waveforms is $M = 2n$, and also $k = \log_2 M$. Hence :

$$\left(\frac{W}{R}\right)_{Had} = \frac{M}{2\log_2 M}$$

which is the same as M-ary FSK.

Problem 8.20 :

From (8-1-47) of the text, the correlation coefficient between the all-zero codeword and the l-th codeword is $\rho_l = 1 - 2w_l/n$, where w_l is the weight of the l-th codeword. For the maximum length shift register codes : $n = 2^m - 1 = M - 1$ (where m is the parameter of the code) and $w_l = 2^{m-1}$ for all codewords except the all-zero codeword. Hence :

$$\rho_l = 1 - \frac{22^{m-1}}{2^m - 1} = -\frac{1}{2^m - 1} = -\frac{1}{M-1}$$

for all l. Since the code is linear if follows that $\rho = -1/(M-1)$ between any pair of codewords. Note : An alternative way to prove the above is to express each codeword in vector form as

$$s_l = \left[\pm\sqrt{\frac{\mathcal{E}}{n}}, \pm\sqrt{\frac{\mathcal{E}}{n}}, ..., \pm\sqrt{\frac{\mathcal{E}}{n}}\right] \quad \text{(n elements in all)}$$

where $\mathcal{E} = n\mathcal{E}_b$ is the energy per codeword and note that any one codeword differs from each other at exactly 2^{m-1} bits and agrees with the other at $2^{m-1} - 1$ bits. Then the correlation coefficient is :

$$Re\left[\rho_{mk}\right] = \frac{s_l \cdot s_k}{|s_l|\,|s_k|} = \frac{\frac{\mathcal{E}}{n}\left(2^{m-1} \cdot 1 + (2^{m-1} - 1) \cdot (-1)\right)}{\frac{\mathcal{E}}{n}n} = -\frac{1}{n} = -\frac{1}{M-1}$$

Problem 8.21 :

We know that the (7,4) Huffman code has $d_{\min} = 3$ and weight distribution (Problem 8.3) : w=0 (1 codeword), w=3 (7 codewords), w=4 (7 codewords), w=7 (1 codeword).

Hence, for soft-decision decoding (8-1-51) :

$$P_M \leq 7Q\left(\sqrt{\frac{24}{7}\gamma_b}\right) + 7Q\left(\sqrt{\frac{32}{7}\gamma_b}\right) + Q\left(\sqrt{8\gamma_b}\right)$$

or a looser bound (8-1-52) :

$$P_M \leq 15Q\left(\sqrt{\frac{24}{7}\gamma_b}\right)$$

For hard-decision decoding (8-1-82):

$$P_M \leq \sum_{m=2}^{7}\binom{7}{m}p^m(1-p)^{7-m} = 1 - \sum_{m=0}^{1}\binom{7}{m}p^m(1-p)^{7-m} = 1 - 7p(1-p)^6 - (1-p)^7$$

where $p = Q\left(\sqrt{2R_c\gamma_b}\right) = Q\left(\sqrt{\frac{8}{7}\gamma_b}\right)$ or (8-1-90) :

$$P_M \leq 7\left[4p(1-p)\right]^{3/2} + 7\left[4p(1-p)\right]^2 + \left[4p(1-p)\right]^{7/2}$$

or (8-1-91) :

$$P_M \leq 14\left[4p(1-p)\right]^{3/2}$$

Problem 8.22 :

We assume that the all-zero codeword is transmitted and we determine the probability that we select codeword C_m having weight w_m. We define a random variable $X_i,\ i = 1, 2, ...w_m$ as :

$$X_i = \left\{ \begin{array}{cc} 1, & \text{with probability p} \\ -1, & \text{with probability 1 − p} \end{array} \right\}$$

where p is the error probability for a bit. Then, we will erroneously select a codeword C_m of weight w_m, if more than $w_m/2$ bits are in error or if $\sum_{i=1}^{w_m} X_i \geq 0$. We assume that $p < 1/2$; then, following the exact same procedure as in Example 2-1-7 (page 60 of the text), we show that :

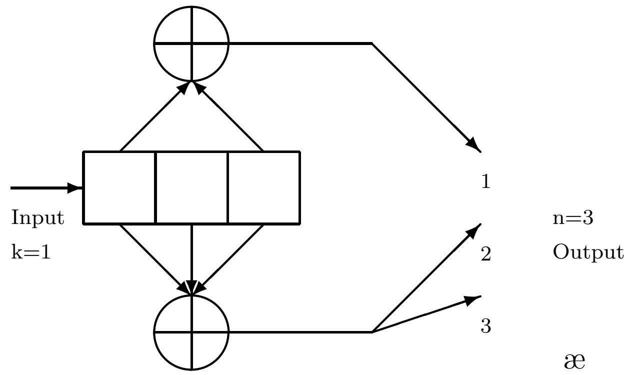
$$P\left(\sum_{i=1}^{w_m} X_i \geq 0\right) \leq \left[4p(1-p)\right]^{w_m/2}$$

By applying the union bound we obtain the desired result :

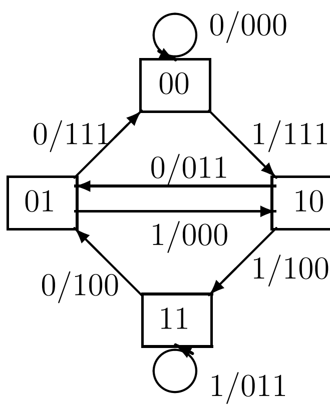
$$P_M \leq \sum_{m=2}^{M}\left[4p(1-p)\right]^{w_m/2}$$

Problem 8.23 :

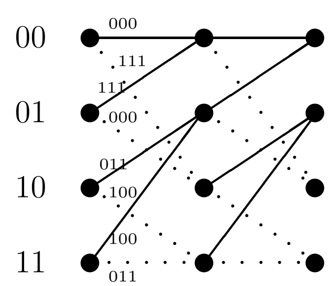
(a) The encoder for the $(3,1)$ convolutional code is depicted in the next figure.



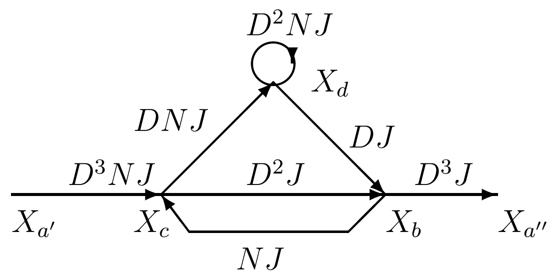
(b) The state transition diagram for this code is depicted in the next figure.



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned}
X_c &= D^3NJX_{a'} + NJX_b \\
X_b &= D^2JX_c + DJX_d \\
X_d &= DNJX_c + D^2NJX_d \\
X_{a''} &= D^3JX_b
\end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^8NJ^3(1+NJ-D^2NJ)}{1-D^2NJ(1+NJ^2+J-D^2J^2)}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

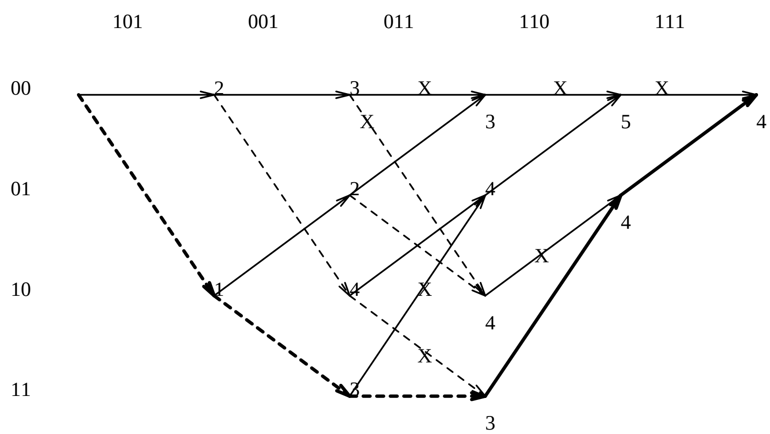
$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^8(1-2D^2)}{1-D^2(3-D^2)} = D^8 + 2D^{10} + \cdots$$

Hence, $d_{\text{free}} = 8$

(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.24 :

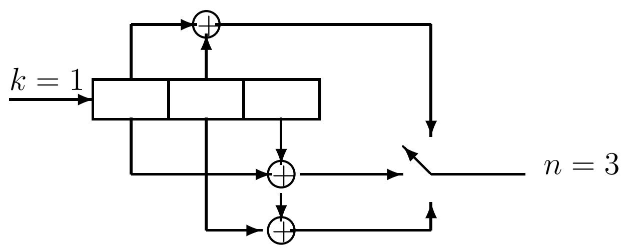
The code of Problem 8-23 is a $(3,1)$ convolutional code with $K = 3$. The length of the received sequence $\mathbf{y}$ is 15. This means that 5 symbols have been transmitted, and since we assume that the information sequence has been padded by two 0's, the actual length of the information sequence is 3. The following figure depicts 5 frames of the trellis used by the Viterbi decoder. The numbers on the nodes denote the metric (Hamming distance) of the survivor paths (the non-survivor paths are shown with an X). In the case of a tie of two merging paths at a node, we have purged the upper path.



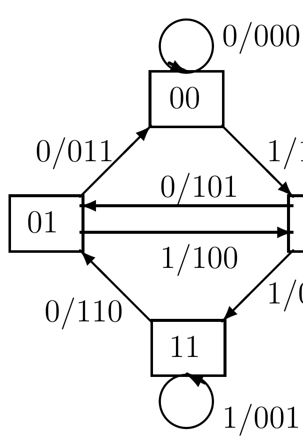
The decoded sequence is $\{111, 100, 011, 100, 111\}$ (i.e the path with the minimum final metric - heavy line) and corresponds to the information sequence $\{1, 1, 1\}$ followed by two zeros.

Problem 8.25 :

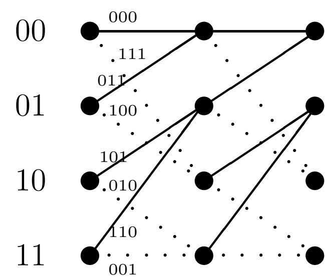
(a) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



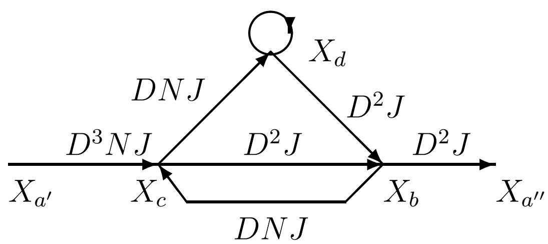
(b) The state transition diagram for this code is shown below



(c) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



(d) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= D^2JX_c + D^2JX_d \\ X_d &= DNJX_c + DNJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

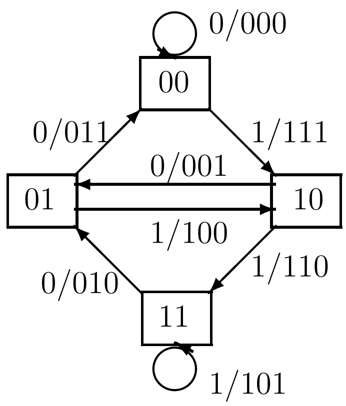
$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

Hence, $d_{\text{free}} = 7$

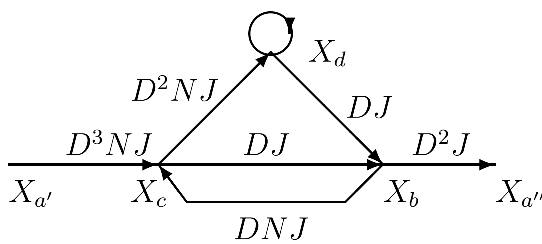
(e) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 8.26 :

(a) The state transition diagram for this code is depicted in the next figure.



(b) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= DJX_c + DJX_d \\ X_d &= D^2NJX_c + D^2NJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

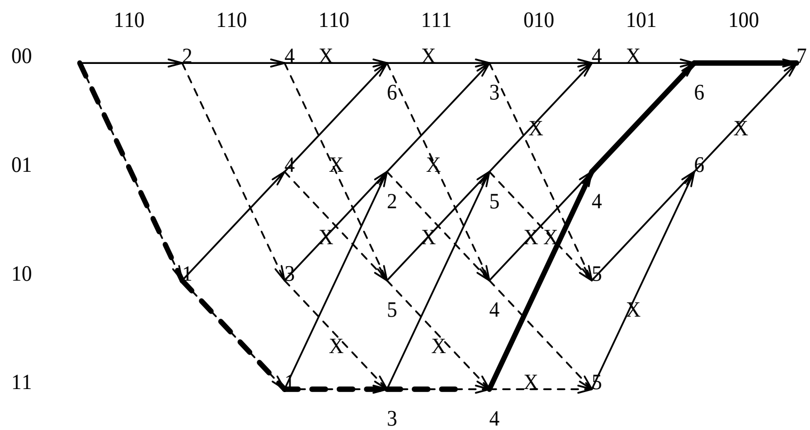
$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6NJ^3}{1 - D^2NJ - D^2NJ^2}$$

(c) To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^{10} + \cdots$$

Hence, $d_{\text{free}} = 6$

(d) The following figure shows 7 frames of the trellis diagram used by the Viterbi decoder. It is assumed that the input sequence is padded by two zeros, so that the actual length of the information sequence is 5. The numbers on the nodes indicate the Hamming distance of the survivor paths. The deleted branches have been marked with an X. In the case of a tie we deleted the upper branch. The survivor path at the end of the decoding is denoted by a thick line.



The information sequence is 11110 and the corresponding codeword 111 110 101 101 010 011 000...

(e) An upper to the bit error probability of the code is given by

$$P_b \leq \left. \frac{dT(D,N,J=1)}{dN} \right|_{N=1, D=\sqrt{4p(1-p)}}$$

But

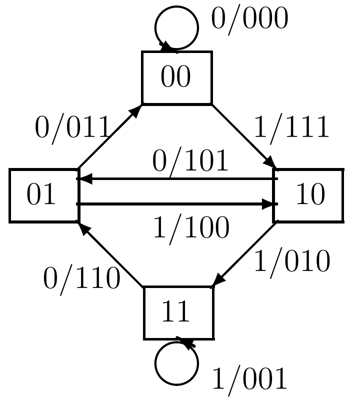
$$\frac{dT(D,N,1)}{dN} = \frac{d}{dN}\left[\frac{D^6 N}{1-2D^2N}\right] = \frac{D^6 - 2D^8(1-N)}{(1-2D^2N)^2}$$

and since $p = 10^{-5}$, we obtain

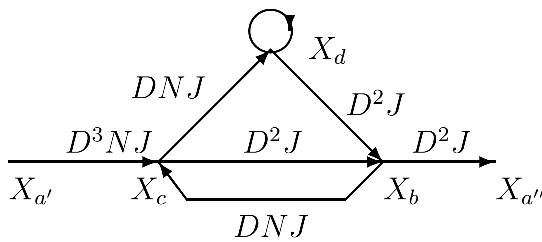
$$P_b \leq \left. \frac{D^6}{(1-2D^2)^2} \right|_{D=\sqrt{4p(1-p)}} \approx 6.14 \cdot 10^{-14}$$

Problem 8.27 :

(a) The state transition diagram for this code is shown below



(b) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= D^2JX_c + D^2JX_d \\ X_d &= DNJX_c + DNJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

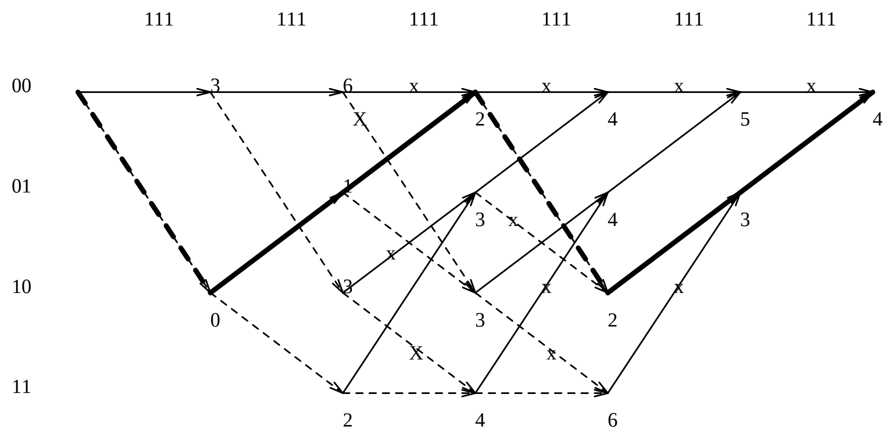
(c) To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

Hence, $d_{\text{free}} = 7$. The path, which is at a distance d_{free} from the all zero path, is the path $X_a \rightarrow X_c \rightarrow X_b \rightarrow X_a$.

(d) The following figure shows 6 frames of the trellis diagram used by the Viterbi algorithm to decode the sequence $\{111, 111, 111, 111, 111, 111\}$. The numbers on the nodes indicate the

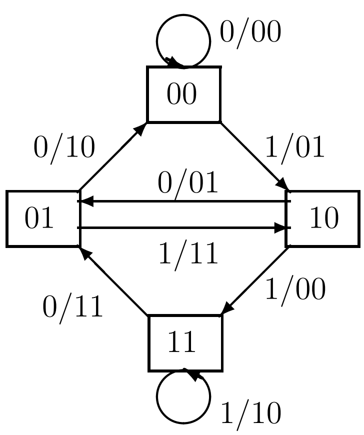
Hamming distance of the survivor paths from the received sequence. The branches that are dropped by the Viterbi algorithm have been marked with an X. In the case of a tie of two merging paths, we delete the upper path.

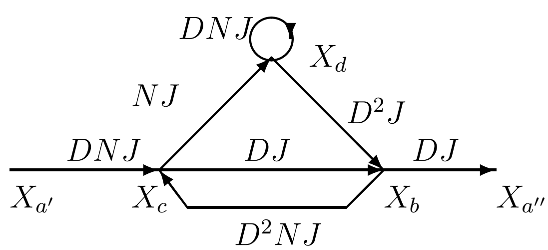


The decoded sequence is $\{111, 101, 011, 111, 101, 011\}$ which coresponds to the information sequence $\{x_1, x_2, x_3, x_4\} = \{1, 0, 0, 1\}$ followed by two zeros.

Problem 8.28 :

(a) The state transition diagram and the flow diagram used to find the transfer function for this code are depicted in the next figure.





Thus,

$$
\begin{aligned}
X_c &= DNJX_{a'} + D^2NJX_b \\
X_b &= DJX_c + D^2JX_d \\
X_d &= NJX_c + DNJX_d
\end{aligned}
$$

$$X_{a''} = DJX_b$$

and by eliminating X_b, X_c and X_d, we obtain

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^3NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the transfer function of the code in the form $T(D,N)$, we set $J = 1$ in $T(D,N,J)$. Hence,

$$T(D,N) = \frac{D^3N}{1 - DN - D^3N}$$

(b) To find the free distance of the code we set $N = 1$ in the transfer function $T(D,N)$, so that

$$T_1(D) = T(D,N)|_{N=1} = \frac{D^3}{1 - D - D^3} = D^3 + D^4 + D^5 + 2D^6 + \cdots$$

Hence, $d_{\text{free}} = 3$

(c) An upper bound on the bit error probability, when hard decision decoding is used, is given by (see (8-2-34))

$$P_b \leq \frac{1}{k}\frac{dT(D,N)}{dN}\bigg|_{N=1, D=\sqrt{4p(1-p)}}$$

Since

$$\frac{dT(D,N)}{dN}\bigg|_{N=1} = \frac{d}{dN}\frac{D^3N}{1-(D+D^3)N}\bigg|_{N=1} = \frac{D^3}{(1-(D+D^3))^2}$$

with $k = 1$, $p = 10^{-6}$ we obtain

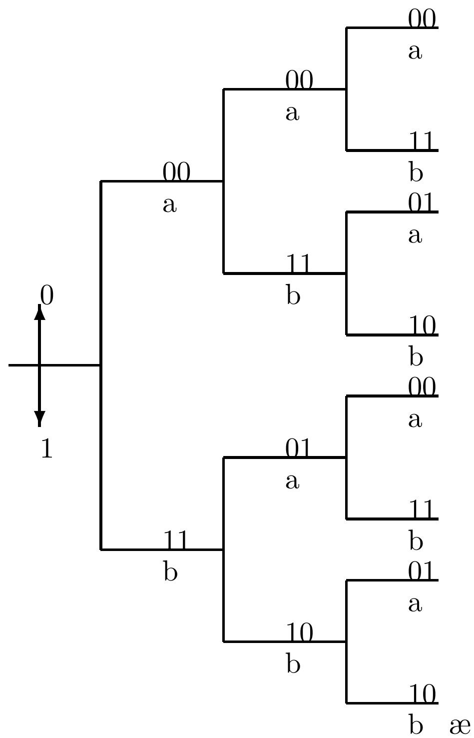
$$P_b \leq \frac{D^3}{(1-(D+D^3))^2}\bigg|_{D=\sqrt{4p(1-p)}} = 8.0321 \times 10^{-9}$$

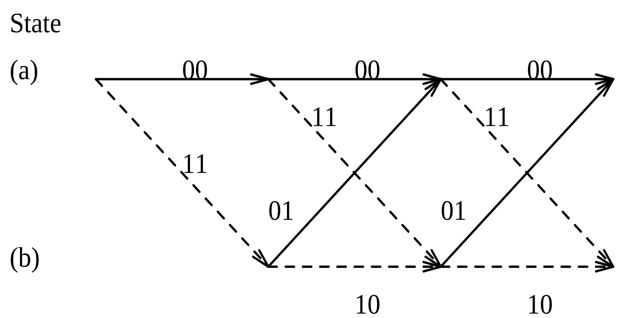
Problem 8.29 :

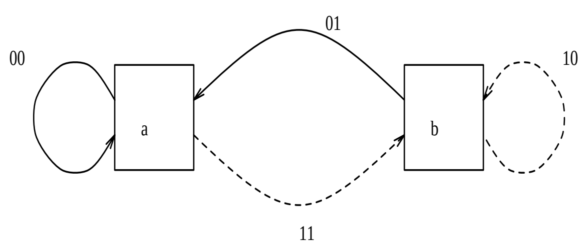
(a)

$$g_1 = [10],\ g_2 = [11], \quad \text{states : (a)} = [0],\ \text{(b)} = [1]$$

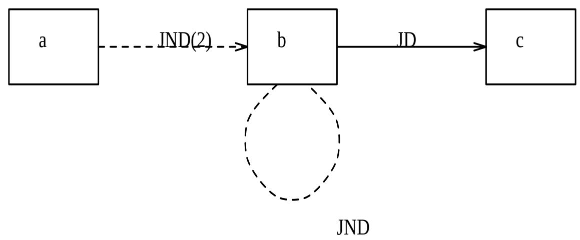
The tree diagram, trellis diagram and state diagram are given in the following figures :







(b) Redrawing the state diagram :



$$X_b = JND^2 X_a + JNDX_b \Rightarrow X_b = \frac{JND^2}{1-JND} X_a$$

$$X_c = JDX_b \Rightarrow \frac{X_c}{X_a} = T(D,N,J) = \frac{J^2ND^3}{1-JND} = J^2ND^3 + J^3N^2D^4 + ...$$

Hence :

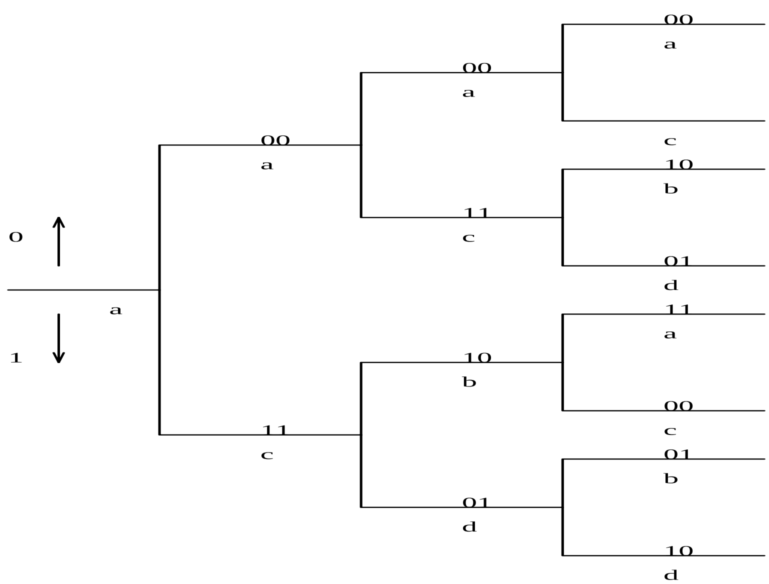
$$d_{\min} = 3$$

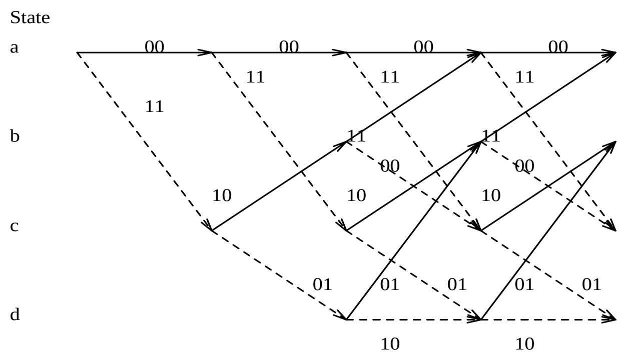
Problem 8.30 :

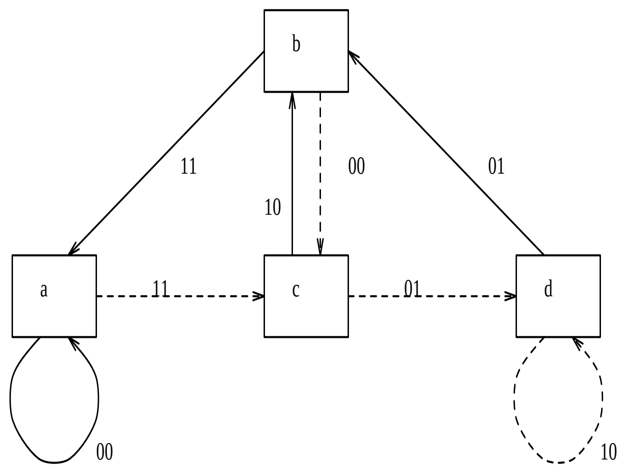
(a)

$$g_1 = [111],\ g_2 = [101], \quad \text{states : (a)} = [00],\ \text{(b)} = [01],\ \text{(c)} = [10],\ \text{(d)} = [11]$$

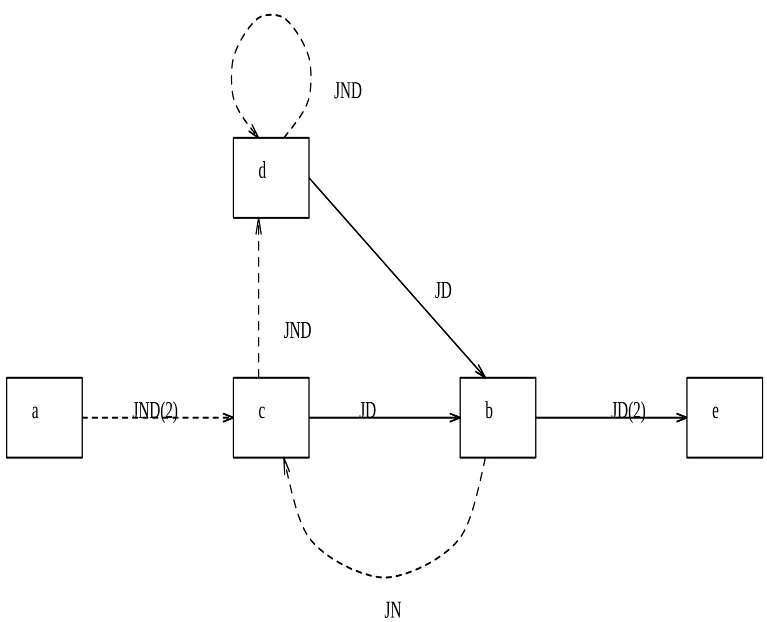
The tree diagram, trellis diagram and state diagram are given in the following figures :







(b) Redrawing the state diagram :



$$\left.\begin{array}{l} X_c = JND^2X_a + JNX_b \\ X_b = JDX_c + JDX_d \\ X_d = JNDX_d + JNDX_c = NX_b \end{array}\right\} \Rightarrow X_b = \frac{J^2ND^3}{1 - JND(1+J)}X_a$$

$$X_e = JD^2X_b \Rightarrow \frac{X_e}{X_a} = T(D,N,J) = \frac{J^3ND^5}{1 - JND(1+J)} = J^3ND^5 + J^4N^2D^6(1+J) + ...$$

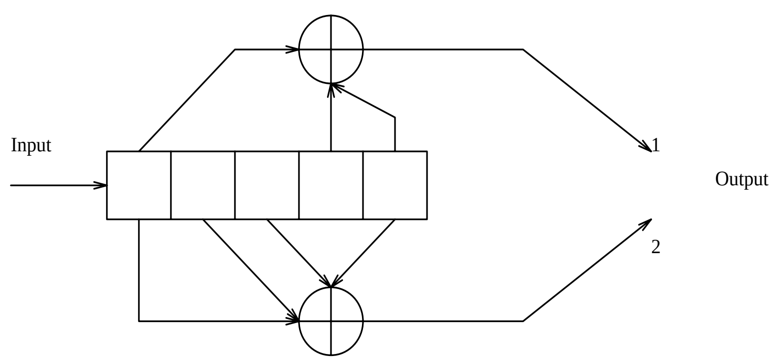
Hence :

$$d_{\min} = 5$$

Problem 8.31 :

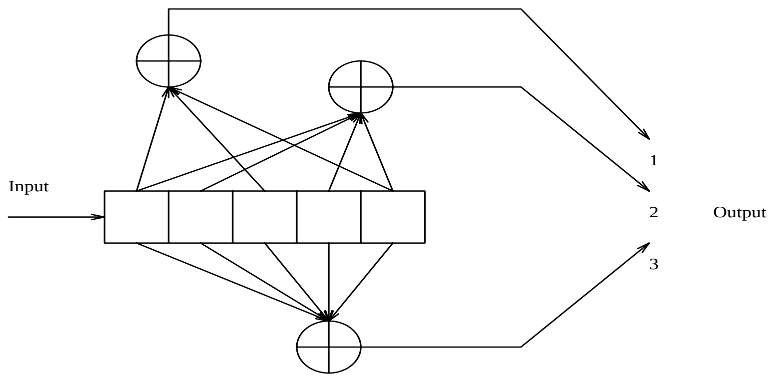
(a)

$$g_1 = [23] = [10011],\ g_2 = [35] = [11101]$$



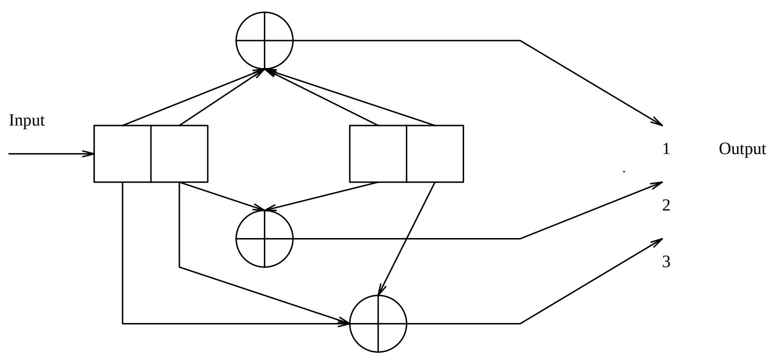
(b)

$$g_1 = [25] = [10101],\ g_2 = [33] = [11011],\ g_3 = [37] = [11111]$$



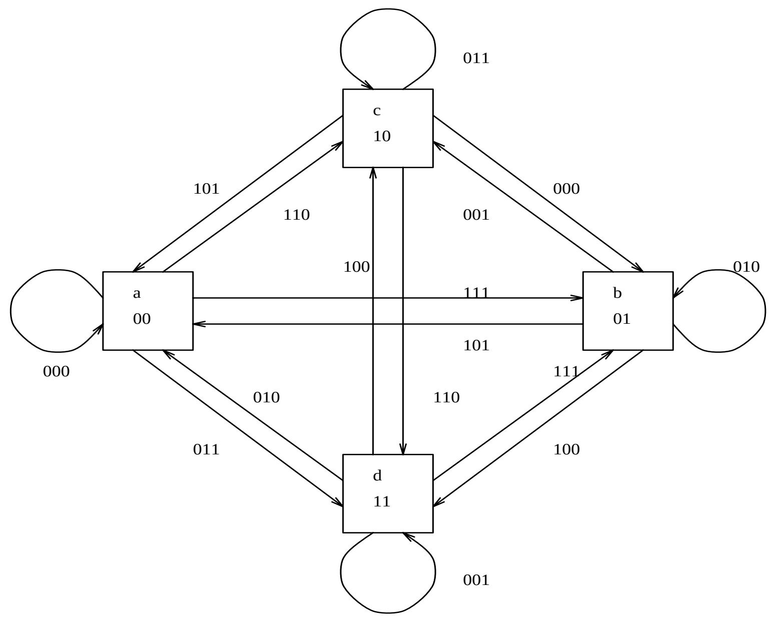
(c)

$$g_1 = [17] = [1111], \; g_2 = [06] = [0110], \; g_3 = [15] = [1101]$$



Problem 8.32 :

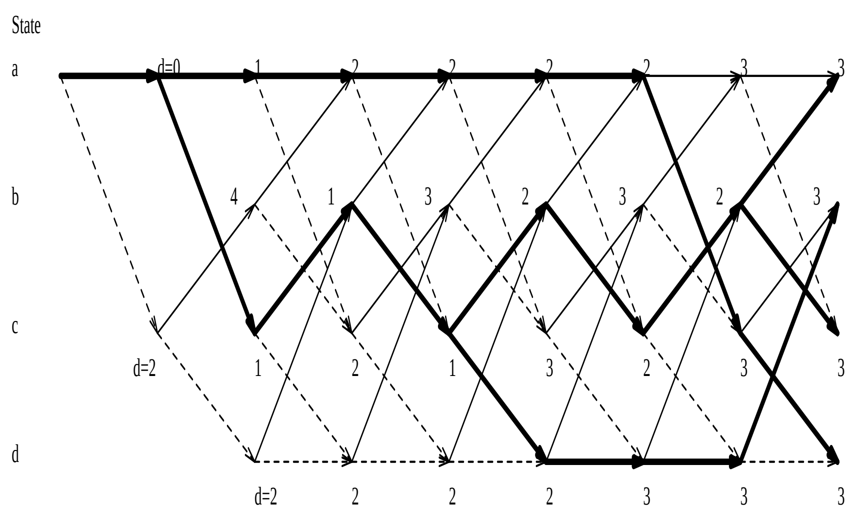
For the encoder of Probl. 8.31(c), the state diagram is as follows :



The 2-bit input that forces the transition from one state to another is the 2-bits that characterize the terminal state.

Problem 8.33 :

The encoder is shown in Probl. 8.30. The channel is binary symmetric and the metric for Viterbi decoding is the Hamming distance. The trellis and the surviving paths are illustrated in the following figure :



Problem 8.34 :

In Probl. 8.30 we found :

$$T(D,N,J) = \frac{J^3 N D^5}{1 - JND(1+J)}$$

Setting $J=1$:

$$T(D,N) = \frac{ND^5}{1-2ND} \Rightarrow \frac{dT(D,N)}{dN} = \frac{D^5}{(1-2ND)^2}$$

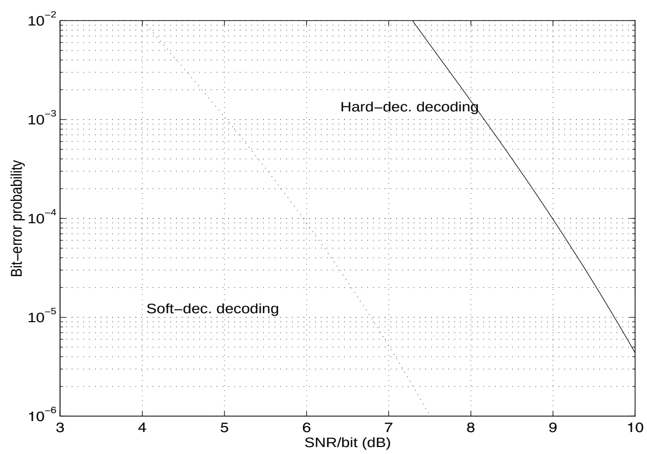
For soft-decision decoding the bit-error probability can be upper-bounded by :

$$P_{bs} \leq \frac{1}{2}\frac{dT(D,N)}{dN}|_{N=1,D=\exp(-\gamma_b R_c)} = \frac{1}{2}\frac{D^5}{(1-2ND)^2}|_{N=1,D=\exp(-\gamma_b/2)} = \frac{1}{2}\frac{\exp(-5\gamma_b/2)}{(1-\exp(-\gamma_b/2))^2}$$

For hard-decision decoding, the Chernoff bound is :

$$P_{bh} \leq \frac{dT(D,N)}{dN}|_{N=1,D=\sqrt{4p(1-p)}} = \frac{\left[\sqrt{4p(1-p)}\right]^{5/2}}{\left[1-2\sqrt{4p(1-p)}\right]^2}$$

where $p = Q\left(\sqrt{\gamma_b R_c}\right) = Q\left(\sqrt{\gamma_b/2}\right)$ (assuming binary PSK). A comparative plot of the bit-error probabilities is given in the following figure :

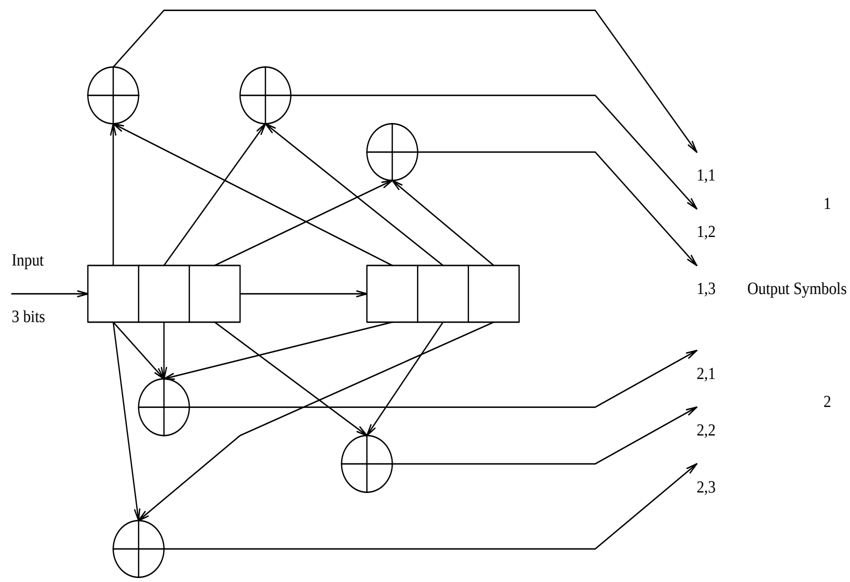


Problem 8.35 :

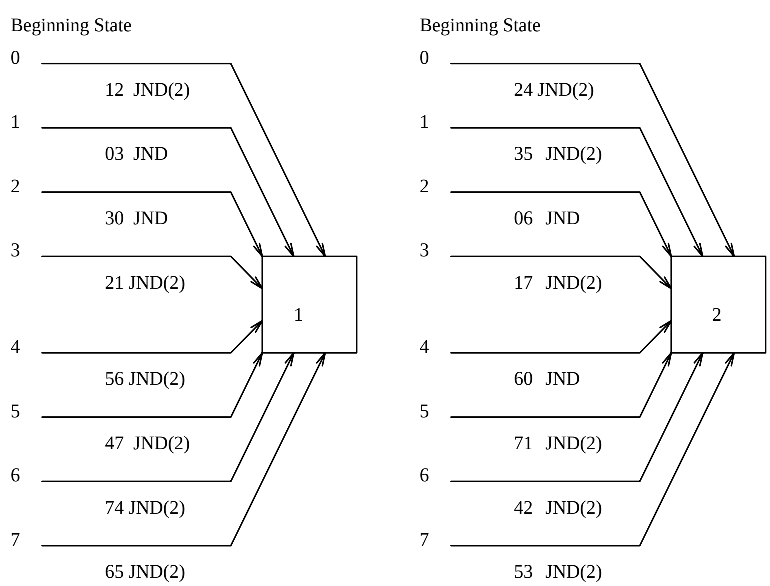
For the dual-3 (k=3), rate 1/2 code, we have from Table 8-2-36 :

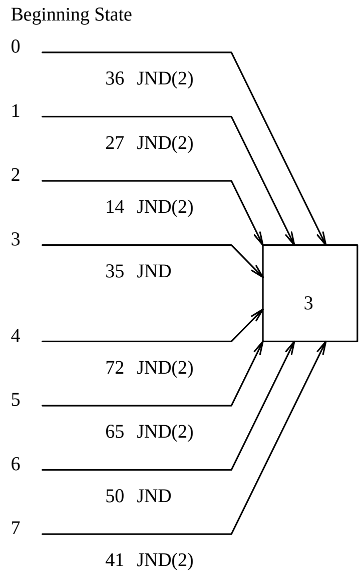
$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 100100 \\ 010010 \\ 001001 \end{bmatrix}, \qquad \begin{bmatrix} g_4 \\ g_5 \\ g_6 \end{bmatrix} = \begin{bmatrix} 110100 \\ 001010 \\ 100001 \end{bmatrix}$$

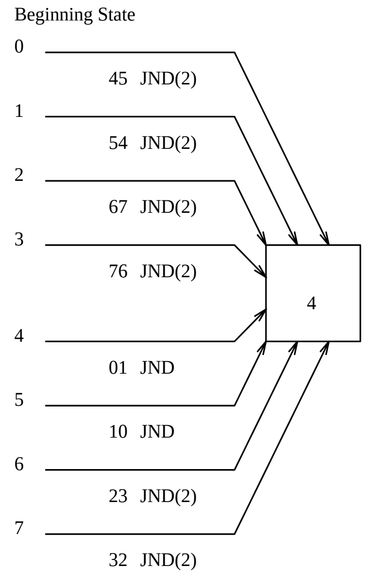
Hence, the encoder will be :

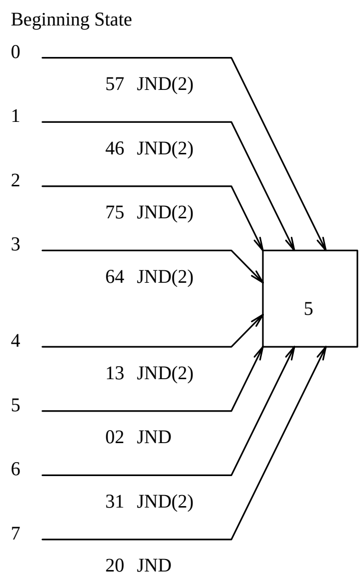


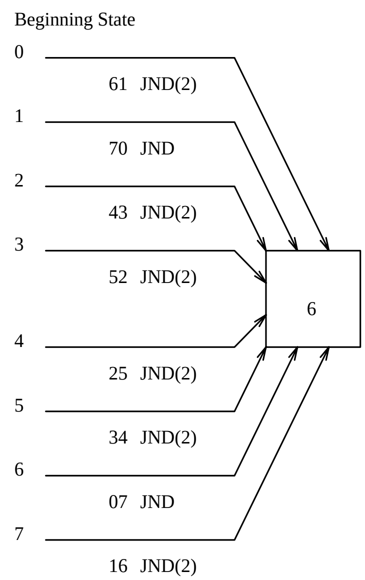
The state transitions are given in the following figures :

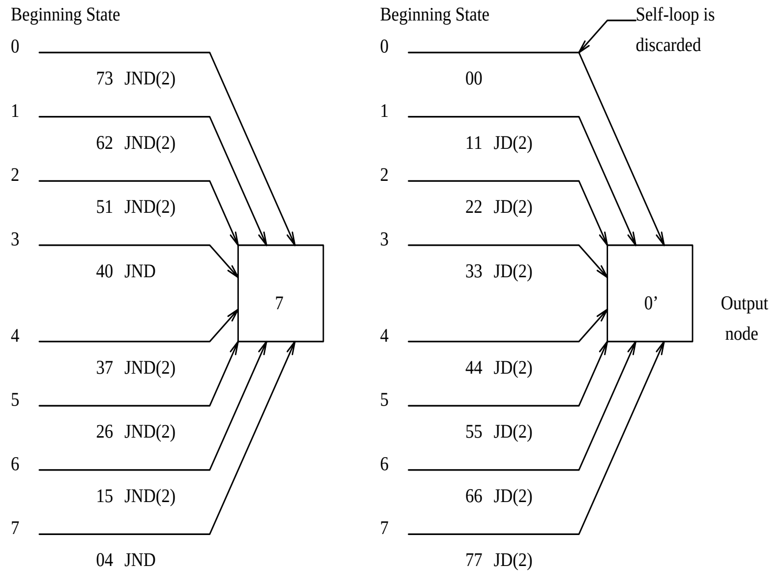



Beginning State
0
36 JND(2)
1
27 JND(2)
2
14 JND(2)
3
35 JND
3
4
72 JND(2)
5
65 JND(2)
6
50 JND
7
41 JND(2)


Beginning State
0
45 JND(2)
1
54 JND(2)
2
67 JND(2)
3
76 JND(2)
4
4
01 JND
5
10 JND
6
23 JND(2)
7
32 JND(2)


Beginning State
0
57 JND(2)
1
46 JND(2)
2
75 JND(2)
3
64 JND(2)
5
4
13 JND(2)
5
02 JND
6
31 JND(2)
7
20 JND


Beginning State
0
61 JND(2)
1
70 JND
2
43 JND(2)
3
52 JND(2)
6
4
25 JND(2)
5
34 JND(2)
6
07 JND
7
16 JND(2)



The states are : $(1) = 000,\ (2) = 001,\ (3) = 010,\ (4) = 011,\ (5) = 100,\ (6) = 101,\ (7) = 110,\ (8) = 111$. The state equations are :

$$\begin{aligned}
X_1 &= D^2NJ\left(X_0 + X_3 + X_4 + X_5 + X_6 + X_7\right) + DNJ\left(X_1 + X_2\right)\\
X_2 &= D^2NJ\left(X_0 + X_1 + X_3 + X_5 + X_6 + X_7\right) + DNJ\left(X_2 + X_4\right)\\
X_3 &= D^2NJ\left(X_0 + X_1 + X_2 + X_4 + X_5 + X_7\right) + DNJ\left(X_3 + X_6\right)\\
X_4 &= D^2NJ\left(X_0 + X_1 + X_2 + X_3 + X_6 + X_7\right) + DNJ\left(X_4 + X_5\right)\\
X_5 &= D^2NJ\left(X_0 + X_1 + X_2 + X_3 + X_4 + X_6\right) + DNJ\left(X_5 + X_7\right)\\
X_6 &= D^2NJ\left(X_0 + X_2 + X_3 + X_4 + X_5 + X_7\right) + DNJ\left(X_1 + X_6\right)\\
X_7 &= D^2NJ\left(X_0 + X_1 + X_2 + X_4 + X_5 + X_6\right) + DNJ\left(X_3 + X_7\right)\\
X_0' &= D^2J\left(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7\right)
\end{aligned}$$

where, note that D, N correspond to symbols and not bits. If we add the first seven equations, we obtain :

$$\sum_{i=1}^{7} = 7D^2NJX_0 + 2DNJ\sum_{i=1}^{7} X_i + 5D^2NJ\sum_{i=1}^{7} X_i$$

Hence :

$$\sum_{i=1}^{7} X_i = \frac{7D^2NJ}{1 - 2DNJ - 5D^2NJ}$$

Substituting the result into the last equation we obtain :

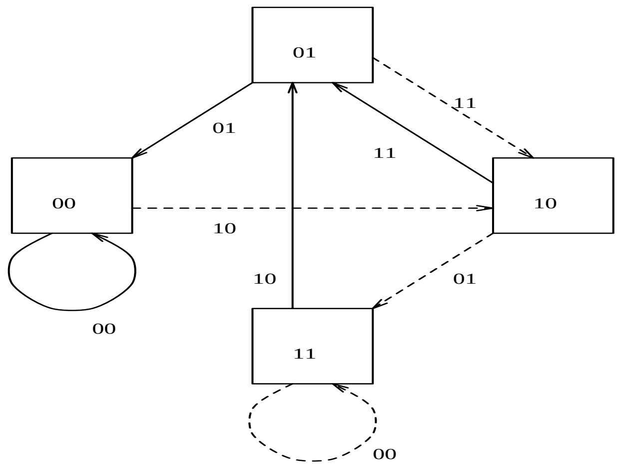
$$\frac{X_0'}{X_0} = T(D, N, J) = \frac{7D^4NJ^2}{1 - 2DNJ - 5D^2NJ} = \frac{7D^4NJ^2}{1 - DNJ(2 + 5D)}$$

which agrees with the result (8-2-37) in the book.

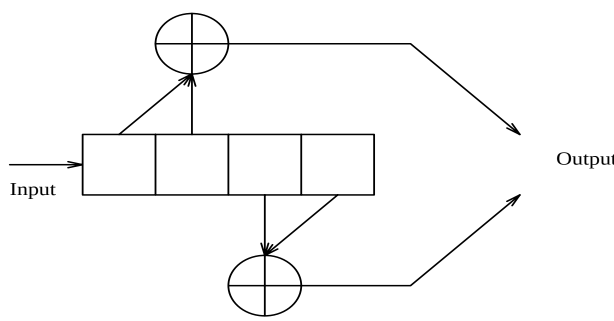
Problem 8.36 :

$$g_1 = [110],\ g_2 = [011], \quad \text{states} \ : \ (a) = [00],\ (b) = [01],\ (c) = [10],\ (d) = [11]$$

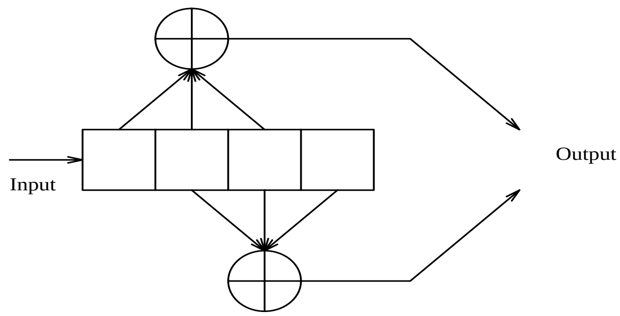
The state diagram is given in the following figure :



We note that this is a catastrophic code, since there is a zero-distance path from a non-zero state back to itself, and this path corresponds to input 1.
A simple example of an $K = 4$, rate 1/2 encoder that exhibits error propagation is the following :



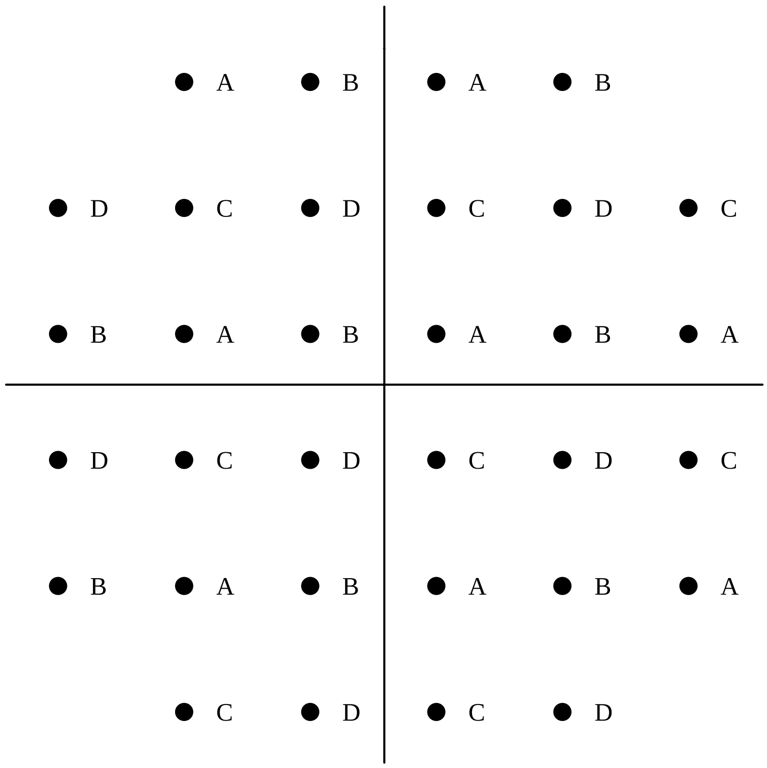
The state diagram for this code has a self-loop in the state 111 with input 1, and output 00.
A more subtle example of a catastrophic code is the following :



In this case there is a zero-distance path generated by the sequence 0110110110..., which encompasses the states 011,101, and 110. that is, if the encoder is in state 011 and the input is 1, the output is 00 and the new state is 101. If the next bit is 1, the output is again 00 and the new state is 110. Then if the next bit is a zero, the output is again 00 and the new state is 011, which is the same state that we started with. Hence, we have a closed path in the state diagram which yields an output that is identical to the output of the all-zero path, but which results from the input sequence 110110110...
For an alternative method for identifying rate 1/n catastrophic codes based on observation of the code generators, please refer to the paper by Massey and Sain (1968).

Problem 8.37 :

There are 4 subsets corresponding to the four possible outputs from the rate 1/2 convolutional encoder. Each subset has eight signal points, one for each of the 3-tuples from the uncoded bits. If we denote the sets as A,B,C,D, the set partitioning is as follows :



The minimum distance between adjacent points in the same subset is doubled.

Problem 8.38 :

(a) Let the decoding rule be that the first codeword is decoded when $\mathbf{y}_i$ is received if

$$p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)$$

The set of $\mathbf{y}_i$ that decode into $\mathbf{x}_1$ is

$$Y_1 = \{\mathbf{y}_i : p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)\}$$

The characteristic function of this set $\chi_1(\mathbf{y}_i)$ is by definition equal to 0 if $\mathbf{y}_i \notin Y_1$ and equal to 1 if $\mathbf{y}_i \in Y_1$. The characteristic function can be bounded as

$$1 - \chi_1(\mathbf{y}_i) \leq \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

This inequality is true if $\chi(\mathbf{y}_i) = 1$ because the right side is nonnegative. It is also true if $\chi(\mathbf{y}_i) = 0$ because in this case $p(\mathbf{y}_i|\mathbf{x}_2) > p(\mathbf{y}_i|\mathbf{x}_1)$ and therefore,

$$1 \leq \frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)} \Longrightarrow 1 \leq \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

Given that the first codeword is sent, then the probability of error is

$$\begin{aligned} P(\text{error}|\mathbf{x}_1) &= \sum_{\mathbf{y}_i \in Y - Y_1} p(\mathbf{y}_i|\mathbf{x}_1) = \sum_{\mathbf{y}_i \in Y} p(\mathbf{y}_i|\mathbf{x}_1)[1 - \chi_1(\mathbf{y}_i)] \\ &\leq \sum_{\mathbf{y}_i \in Y} p(\mathbf{y}_i|\mathbf{x}_1) \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)} \right)^{\frac{1}{2}} = \sum_{\mathbf{y}_i \in Y} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} \\ &= \sum_{i=1}^{2^n} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} \end{aligned}$$

where Y denotes the set of all possible sequences $\mathbf{y}_i$. Since, each element of the vector $\mathbf{y}_i$ can take two values, the cardinality of the set Y is 2^n.

(b) Using the results of the previous part we have

$$\begin{aligned} P(\text{error}) &\leq \sum_{i=1}^{2^n} \sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} = \sum_{i=1}^{2^n} p(\mathbf{y}_i) \sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_1)}{p(\mathbf{y}_i)}} \sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i)}} \\ &= \sum_{i=1}^{2^n} p(\mathbf{y}_i) \sqrt{\frac{p(\mathbf{x}_1|\mathbf{y}_i)}{p(\mathbf{x}_1)}} \sqrt{\frac{p(\mathbf{x}_2|\mathbf{y}_i)}{p(\mathbf{x}_2)}} = \sum_{i=1}^{2^n} 2p(\mathbf{y}_i) \sqrt{p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i)} \end{aligned}$$

However, given the vector $\mathbf{y}_i$, the probability of error depends only on those values that $\mathbf{x}_1$ and $\mathbf{x}_2$ are different. In other words, if $x_{1,k} = x_{2,k}$, then no matter what value is the k^{th} element of $\mathbf{y}_i$, it will not produce an error. Thus, if by d we denote the Hamming distance between $\mathbf{x}_1$ and $\mathbf{x}_2$, then

$$p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i) = p^d(1-p)^d$$

and since $p(\mathbf{y}_i) = \frac{1}{2^n}$, we obtain

$$P(\text{error}) = P(d) = 2p^{\frac{d}{2}}(1-p)^{\frac{d}{2}} = [4p(1-p)]^{\frac{d}{2}}$$

Problem 8.39 :

Over P frames, the number of information bits that are being encoded is

$$k_P = P \sum_{j=1}^{J} N_J$$

The number of bits that are being transmitted is determined as follows: For a particular group of bits j, $j = 1, ..., J$, we may delete, with the corresponding puncturing matrix, x_j out of nP bits, on the average, where x may take the values $x = 0, 1, ...(n-1)P - 1$. Remembering that each frame contains N_j bits of the particular group, we arrive at the total average number of bits for each group

$$n(j) = N_j(nP - x_j) \Rightarrow n(j) = N_j(P + M_j), \quad M_j = 1, 2, ..., (n-1)P$$

In the last group $j = J$ we should also add the $K - 1$ overhead information bits, that will add up another $(K-1)(P + M_J)$ transmitted bits to the total average number of bits for the J^{th} group.
Hence, the total number of bits transmitted over P frames be

$$n_P = (K-1)(P + M_J) + \sum_{j=1} J N_j (P + M_j)$$

and the average effective rate of this scheme will be

$$R_{av} = \frac{k_P}{n_P} = \frac{\sum_{j=1}^{J} N_J P}{\sum_{j=1} J N_j (P + M_j) + (K-1)(P + M_J)}$$

Problem 8.39 :

Over P frames, the number of information bits that are being encoded is

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The number of bits that are being transmitted is determined as follows: For a particular group of bits j, $j = 1, ..., J$, we may delete, with the corresponding puncturing matrix, x_j out of nP bits, on the average, where x may take the values $x = 0, 1, ...(n-1)P - 1$. Remembering that each frame contains N_j bits of the particular group, we arrive at the total average number of bits for each group

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and the average effective rate of this scheme will be

$$R_{av} = \frac{k_P}{n_P} = \frac{\sum_{j=1}^{J} N_J P}{\sum_{j=1} J N_j (P + M_j) + (K-1)(P + M_J)}$$

CHAPTER 9

Problem 9.1 :

We want $y(t) = Kx(t - t_0)$. Then :

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$
$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt = K \exp(-j2\pi f t_0) X(f)$$

Therefore :

$$A(f) e^{-j\theta(f)} = K e^{-j2\pi f t_0} \Rightarrow \left\{ \begin{array}{cc} A(f) = K, & \text{for all f} \\ \theta(f) = 2\pi f t_0 \pm n\pi, & n = 0, 1, 2, ... \end{array} \right\}$$

Note that $n\pi$, n odd, results in a sign inversion of the signal.

Problem 9.2 :

(a) Since $\cos(a + \pi/2) = -\sin(a)$, we can write :

$$X(f) = \left\{ \begin{array}{cc} T, & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left[1 - \sin \frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right)\right], & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \end{array} \right\}$$

Then, taking the first two derivatives with respect to f :

$$X'(f) = \left\{ \begin{array}{cc} -\frac{T^2\pi}{2\beta} \cos \frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right), & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{array} \right\}$$

and :

$$X''(f) = \left\{ \begin{array}{cc} \frac{T^3\pi^2}{2\beta^2} \sin \frac{\pi T}{\beta}\left(f - \frac{1}{2T}\right), & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{array} \right\}$$

Therefore the second derivative can be expressed as :

$$X''(f) = -\frac{\pi^2 T^2}{\beta^2}\left[X(f) - \frac{T}{2}\text{rect}\left(\frac{1-\beta}{2T} f\right) - \frac{T}{2}\text{rect}\left(\frac{1+\beta}{2T} f\right)\right]$$

where :

$$\text{rect}(af) = \left\{ \begin{array}{cc} 1, & |f| \le a \\ 0, & \text{o.w} \end{array} \right\}$$

Since the Fourier transform of dx/dt is $j2\pi f X(f)$, we exploit the duality between (f, t), take the inverse Fourier transform of $X''(f)$ and obtain :

$$-4\pi^2 t^2 x(t) = -\frac{\pi^2 T^2}{\beta^2}\left[x(t) - \frac{T}{2}\frac{1}{\pi t} \sin \frac{1-\beta}{2T} 2\pi t - -\frac{T}{2}\frac{1}{\pi t} \sin \frac{1+\beta}{2T} 2\pi t\right]$$

Solving for $x(t)$ we obtain :

$$\begin{aligned} x(t) &= \frac{1}{1-4\beta^2t^2/T}\left[\frac{1}{2\pi t/T}\left(\sin\frac{1-\beta}{2T}2\pi t + \sin\frac{1+\beta}{2T}2\pi t\right)\right] \\ &= \frac{1}{1-4\beta^2t^2/T}\left[\frac{1}{\pi t/T}\left(\sin\frac{\pi t}{T}\cos\frac{\pi\beta t}{T}\right)\right] \end{aligned}$$

(b) When $\beta = 1,\ X(f)$ is non-zero in $|f| \leq 1/T$, and :

$$X(f) = \frac{T}{2}\left(1 + \cos\pi T f\right)$$

The Hilbert transform is :

$$\hat{X}(f) = \left\{ \begin{array}{ll} -j\frac{T}{2}\left(1+\cos\pi T f\right), & 0 \leq f \leq 1/T \\ j\frac{T}{2}\left(1+\cos\pi T f\right), & -1/T \leq f \leq 0 \end{array} \right\}$$

Then :

$$\begin{aligned} \hat{x}(t) &= \int_{-\infty}^{\infty}\hat{X}(f)\exp(j2\pi f t)dt \\ &= \int_{-1/T}^{0}\hat{X}(f)\exp(j2\pi f t)dt + \int_0^{1/T}\hat{X}(f)\exp(j2\pi f t)dt \end{aligned}$$

Direct substitution for $\hat{X}(f)$ yields the result :

$$\hat{x}(t) = \frac{T}{\pi t}\left[\frac{\sin^2\pi t/T - 4t^2/T^2}{1-4t^2/T^2}\right]$$

Note that $\hat{x}(t)$ is an odd function of t.

(c) No, since $\hat{x}(0) = 0$ and $\hat{x}(nT) \neq 0$, for $n \neq 0$. Also $\sum_{n=-\infty}^{\infty}\hat{X}(f+n/2T) \neq$ constant for $|f| \leq 1/2T$.

(d) The single-sideband signal is :

$$x(t)\cos 2\pi f_c t \pm \hat{x}(t)\sin 2\pi f_c t = Re\left[\left(x(t) \pm j\hat{x}(t)\right)e^{j2\pi f_c t}\right]$$

The envelope is $a(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$. For β= 1 :

$$a(t) = \frac{1}{\pi t/T}\frac{1}{1-4t^2/T^2}\sqrt{(1-8t^2/T^2)\sin^2(\pi t/T) + 16t^4/T^4}$$

Problem 9.3 :

(a) $\sum_k h(t-kT) = u(t)$ is a periodic signal with period T. Hence, $u(t)$ can be expanded in the Fourier series :

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{j2\pi n t/T}$$

where :

$$\begin{aligned} u_n &= \frac{1}{T}\int_{-T/2}^{T/2} u(t)\exp(-j2\pi nt/T)dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} h(t-kT)\exp(-j2\pi nt/T)dt \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} h(t-kT)\exp(-j2\pi nt/T)dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} h(t)\exp(-j2\pi nt/T)dt = \frac{1}{T}H\left(\frac{n}{T}\right) \end{aligned}$$

Then : $u(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)e^{j2\pi nt/T} \Rightarrow U(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)\delta\left(f-\frac{n}{T}\right)$. Since $x(t) = u(t)g(t)$, it follows that $X(f) = U(t) * G(f)$. Hence :

$$X(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)G\left(f-\frac{n}{T}\right)$$

(b)
(i)

$$\sum_{k=-\infty}^{\infty} h(kT) = u(0) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)$$

(ii)

$$\sum_{k=-\infty}^{\infty} h(t-kT) = u(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H\left(\frac{n}{T}\right)e^{j2\pi nt/T}$$

(iii) Let

$$v(t) = h(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} h(kT)\delta(t-kT)$$

Hence :

$$V(f) = \sum_{k=-\infty}^{\infty} h(kT)e^{-j2\pi fkT}$$

But

$$\begin{aligned} V(f) &= H(f) * \text{Fourier transform of } \sum_{k=-\infty}^{\infty}\delta(t-kT) \\ &= H(f) * \frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} H(f-\frac{n}{T}) \end{aligned}$$

(c) The criterion for no intersymbol interference is $\{h(kT) = 0,\ k \neq 0 \text{ and } h(0) = 1\}$. If the above condition holds, then from (iii) above we have :

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} H(f-\frac{n}{T}) = \sum_{k=-\infty}^{\infty} h(kT)e^{-j2\pi fkT} = 1$$

Conversely, if $\frac{1}{T}\sum_{n=-\infty}^{\infty} H(f-\frac{n}{T}) = 1,\ \forall f \Rightarrow \sum_{k=-\infty}^{\infty} h(kT)e^{-j2\pi fkT} = 1,\ \forall f$. This is possible only if the left-hand side has no dependence on f, which means $h(kT) = 0,$ for $k \neq 0$. Then $\sum_{k=-\infty}^{\infty} h(kT)e^{-j2\pi fkT} = h(0) = 1$.

Problem 9.4 :

$$x(t) = e^{-\pi a^2 t^2} \Rightarrow X(f) = \frac{1}{a} e^{-\pi f^2/a^2}$$

Hence :

$$X(0) = \frac{1}{a}, \; X(W) = \frac{1}{a} e^{-\pi W^2/a^2}$$

We have :

$$\frac{X(W)}{X(0)} = 0.01 \Rightarrow e^{-\pi W^2/a^2} = 0.01 \Rightarrow W^2 = -\frac{a^2}{\pi} \ln(0.01)$$

But due to the condition for the reduced ISI :

$$x(T) = e^{-\pi a^2 T^2} = 0.01 \Rightarrow T^2 = -\frac{1}{\pi a^2} \ln(0.01)$$

Hence $WT = \frac{-1}{\pi} \ln(0.01) = 1.466$ or :

$$W = \frac{1.466}{T}$$

For the raised cosine spectral characteristic (with roll-off factor 1) $W = 1/T$. Hence, the Gaussian shaped pulse requires more bandwidth than the pulse having the raised cosine spectrum.

Problem 9.5 :

The impulse response of a square-root raised cosine filter is given by

$$x_{ST}(t) = \int_{-\frac{1+\beta}{2T}}^{\frac{1+\beta}{2T}} \sqrt{X_{rc}(f)} e^{j2\pi ft} df$$

where $X_{rc}(f)$ is given by (9.2-26). Splitting the integral in three parts we obtain

$$\begin{aligned} x_{ST}(t) &= \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \sqrt{T/2} \sqrt{1 + \cos\left(\frac{\pi T}{\beta}(-f - \frac{1-\beta}{2T})\right)} e^{j2\pi ft} df \qquad (1) \\ &+ \int_{-\frac{1-\beta}{2T}}^{\frac{1-\beta}{2T}} \sqrt{T} e^{j2\pi ft} df \qquad (2) \\ &+ \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \sqrt{T/2} \sqrt{1 + \cos\left(\frac{\pi T}{\beta}(f - \frac{1-\beta}{2T})\right)} e^{j2\pi ft} df \qquad (3) \end{aligned}$$

The second term (2) gives immediately

$$(2) = \frac{\sqrt{T}}{\pi t} \sin(\pi(1-\beta)t/T)$$

The third term can be solved with the transformation $\lambda = f - \frac{1-\beta}{2T}$. Then

$$(3) = \int_0^{\frac{\beta}{T}} \sqrt{T/2}\sqrt{1 + \cos\left(\frac{\pi T\lambda}{\beta}\right)} e^{j2\pi t(\lambda + \frac{1+\beta}{2T})} d\lambda$$

Using the relationship $1 + \cos 2A = 2\cos^2 A \Rightarrow \sqrt{1+\cos 2A} = \sqrt{2}|\cos A| = \sqrt{2}\cos A$, we can rewrite the above expression as

$$(3) = \int_0^{\frac{\beta}{T}} \sqrt{T}\cos\left(\frac{\pi T\lambda}{2\beta}\right) e^{j2\pi t(\lambda + \frac{1+\beta}{2T})} d\lambda$$

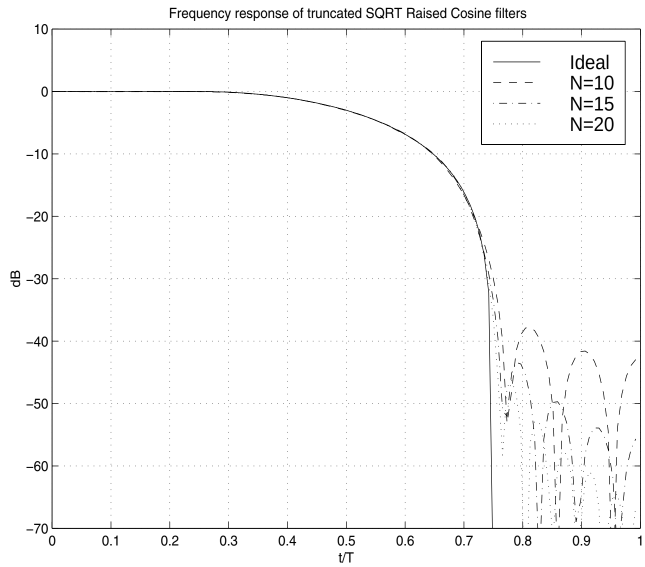
Since $\cos A = \frac{e^{jA} + e^{-jA}}{2}$, the above integral simplifies to the sum of two simple exponential argument intregrals.
Similarly to (3), the first term (1) can be solved with the transformation $\lambda = f + \frac{1-\beta}{2T}$ (notice that $\cos(\frac{\pi T}{\beta}(-f - \frac{1-\beta}{2T})) = \cos(\frac{\pi T}{\beta}(f + \frac{1-\beta}{2T})))$. Then again, the integral simplifies to the sum of two simple exponential argument integrals. Proceeding with adding (1),(2),(3) we arrive at the desired result.

Problem 9.6 :

(a)(b) In order to calculate the frequency response based on the impulse response, we need the values of the impulse response at $t = 0, \pm T/2$, which are not given directly by the expression of Problem 9.5. Using L'Hospital's rule it is straightforward to show that:

$$x(0) = \frac{1}{2} + \frac{2}{\pi}, \quad x(\pm T/2) = \frac{\sqrt{2}}{2}\frac{(2+\pi)}{2\pi}$$

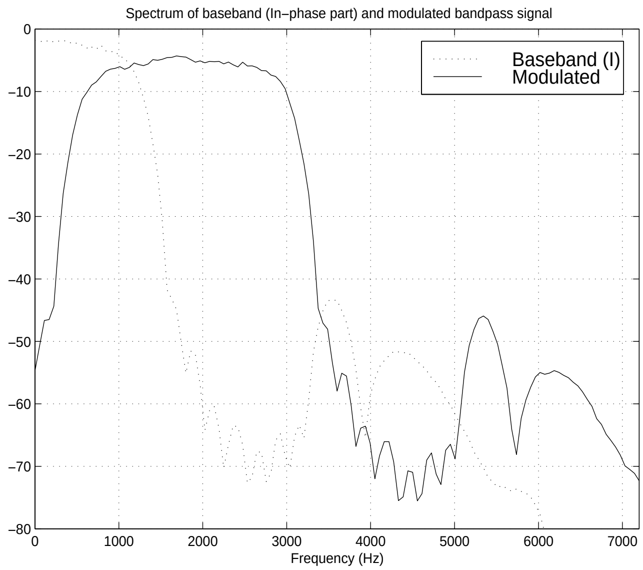
Then, the frequency response of the filters with $N = 10, 15, 20$ compared to the frequency response of the ideal square-root raised cosine filter are depicted in the following figure.



As we see, there is no significant difference in the passband area of the filters, but the realizable, truncated filters do have spectral sidelobes outside their $(1+\beta)/T$ nominal bandwidth. Still, depending on how much residual ISI an application can tolerate, even the $N=10$ filter appears an acceptable approximation of the ideal (non-realizable) square-root raised cosine filter.

Problem 9.7 :

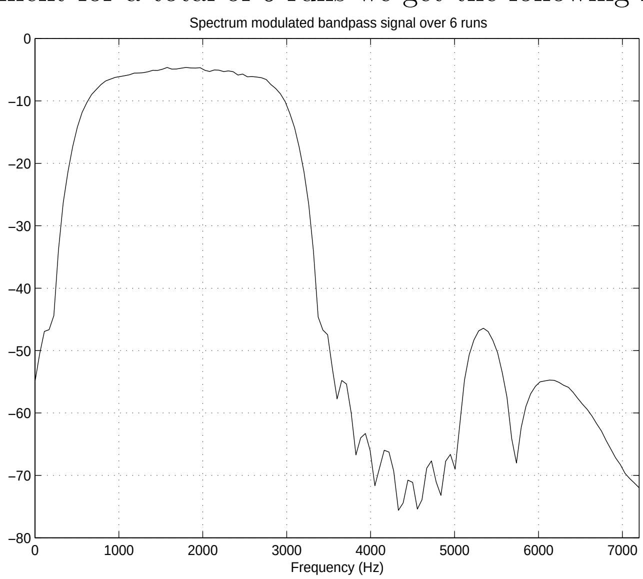
(a),(b) Given a mathematical package like MATLAB, the implementation in software of the digital modulator of Fig P9.7 is relatively straightforward. One comment is that the interpolating filters should have a nominal passband of $[-\pi/3, \pi/3]$, since the interpolation factor applied to the samples at the output of the shaping filter is 3. We chose our interpolation filters (designed with the MATLAB `fir1` function) to have a cutoff frequency (-3 dB frequency) of $\pi/5$. This corresponds to the highest frequency with significant signal content, since with the spectrum of the baseband signal should be (approximately, due to truncation effects) limited to $(1+0.25)/2T$, so samled at $6T$ it should be limited to a discrete frequency of $(2*\pi*(1+0.25)/2T)/6 \approx 0.21*\pi$. The plot with the power spectrum of the digital signal sequence is given in the following figure. We have also plotted the power spectrum of the baseband in-phase (I component) of the signal.



We notice the rather significant sidelobe that is due to the non-completely eliminated image of the spectrum that was generated by the interpolating process. We could mitigate it by choosing an interpolation filter with lower cut-off frequency, but then, we would lose a larger portion of the useful signal as well. The best solution would be to use a longer interpolation filter.

(c)

By repeating the experiment for a total of 6 runs we get the following figure



We notice the smoother shape of the PSD, and we can verify that indeed the spectrum is centered around 1800 Hz.

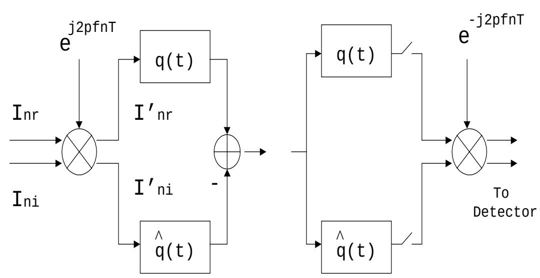
Problem 9.8 :

(a) The alternative expression for $s(t)$ can be rewritten as

$$\begin{aligned}
s(t) \quad ?= & \ \Re\left\{\sum_n I'_n Q(t-nT)\right\} \\
= & \ \Re\left\{\sum_n I_n e^{j2\pi f_c nT} g(t-nT)[\cos 2\pi f_c(t-nT) + j\sin(2\pi f_c(t-nT))]\right\} \\
= & \ \Re\left\{\sum_n I_n g(t-nT)[\cos 2\pi f_c nT + j\sin 2\pi f_c nT][\cos 2\pi f_c(t-nT) + j\sin(2\pi f_c(t-nT))]\right\} \\
= & \ \Re\left\{\sum_n I_n g(t-nT)[\cos 2\pi f_c nT \cos 2\pi f_c(t-nT) - \sin 2\pi f_c nT \sin 2\pi f_c(t-nT)\right. \\
& \left. + j\sin 2\pi f_c nT \cos 2\pi f_c(t-nT) + j\cos 2\pi f_c nT \sin 2\pi f_c(t-nT)]\right\} \\
= & \ \Re\left\{\sum_n I_n g(t-nT)[\cos 2\pi f_c t + j\sin 2\pi f_c t]\right\} \\
= & \ \Re\left\{\sum_n I_n g(t-nT) e^{2\pi f_c t}\right\} = s(t)
\end{aligned}$$

So, indeed the alternative expression for $s(t)$ is a valid one.

(b)



Modulator
(with phase rotator)

Demodulator
(with phase derotator)

Problem 9.9 :

(a) From the impulse response of the pulse having a square-root raised cosine characteristic, which is given in problem 9.5, we can see immediately that $x_{SQ}(t) = x_{SQ}(-t)$, i.e. the pulse $g(t)$ is an even function. We know that the product of an even function times and even function has even symmetry, while the product of even times odd has odd symmetry. Hence $q(t)$ is even, while $\hat{q}(t)$ is odd. Hence, the product $q(t)\hat{q}(t)$ has odd symmetry. We know that the (symmetric around 0) integral of an odd function is zero, or

$$\int_{-\infty}^{\infty} q(t)\hat{q}(t)dt = \int_{-(1+\beta)/2T}^{(1+\beta)/2T} q(t)\hat{q}(t)dt = 0$$

(b) We notice that when $f_c = k/T$, where k is an integer, then the rotator/derotaror of a carrierless QAM system (described in Problem 9.8) gives a trivial rotation of an integer number of full circles ($2\pi kn$), and the carrierless QAM/PSK is equivalent to CAP.

Problem 9.10 :

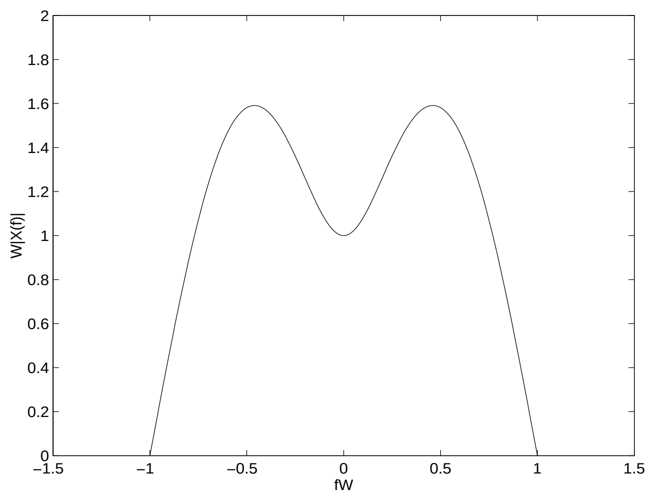
(a)
(i) $x_0 = 2,\ x_1 = 1,\ x_2 = -1$, otherwise $x_n = 0$. Then :

$$x(t) = 2\frac{\sin(2\pi Wt)}{2\pi Wt} + \frac{\sin(2\pi W(t-1/2W))}{2\pi W(t-1/2W)} - \frac{\sin(2\pi W(t-1/W))}{2\pi W(t-1/W)}$$

and :

$$X(f) = \tfrac{1}{2W}\left[2 + e^{-j\pi f/W} - e^{-j2\pi f/W}\right], \quad |f| \leq W \Rightarrow$$
$$|X(f)| = \tfrac{1}{2W}\left[6 + 2\cos\tfrac{\pi f}{W} - 4\cos\tfrac{2\pi f}{W}\right]^{1/2}, \quad |f| \leq W$$

The plot of $|X(f)|$ is given in the following figure :



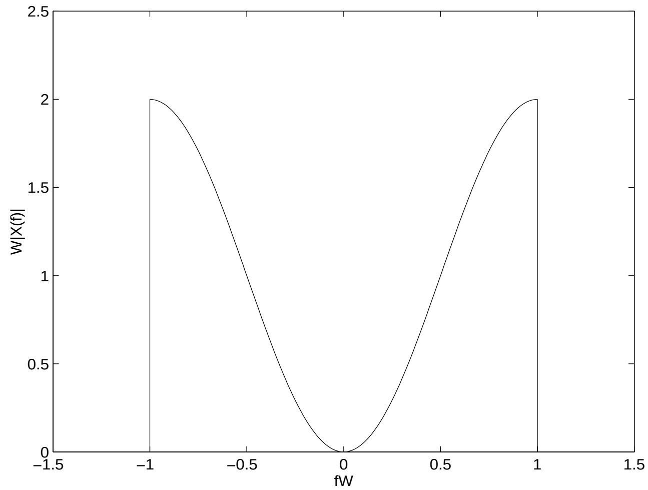
(ii) $x_{-1} = -1,\ x_0 = 2, x_1 = -1$, otherwise $x_n = 0$. Then :

$$x(t) = 2\frac{\sin(2\pi Wt)}{2\pi Wt} - \frac{\sin(2\pi W(t+1/2W))}{2\pi W(t+1/2W)} - \frac{\sin(2\pi W(t-1/2W))}{2\pi W(t-1/2W)}$$

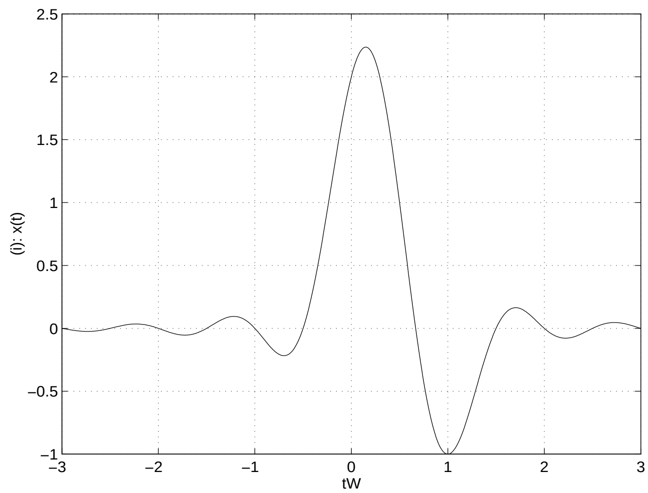
and :

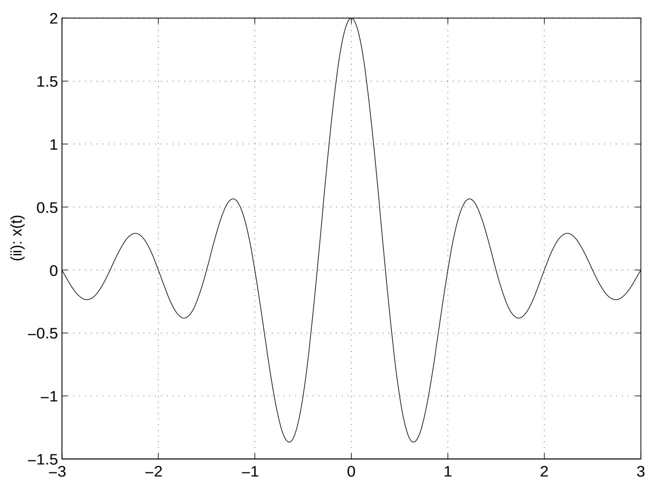
$$X(f) = \frac{1}{2W}\left[2 - e^{-j\pi f/W} - e^{+j\pi f/W}\right] = \frac{1}{2W}\left[2 - 2\cos\frac{\pi f}{W}\right] = \frac{1}{W}\left[1 - \cos\frac{\pi f}{W}\right],\ |f| \leq W$$

The plot of $|X(f)|$ is given in the following figure :



(b) Based on the results obtained in part (a) :





(c) The possible received levels at the receiver are given by :
(i)

$$B_n = 2I_n + I_{n-1} - I_{n-2}$$

where $I_m = \pm 1$. Hence :

$$\begin{array}{c} P(B_n = 0) = 1/4 \\ P(B_n = -2) = 1/4 \\ P(B_n = 2) = 1/4 \\ P(B_n = -4) = 1/8 \\ P(B_n = 4) = 1/8 \end{array}$$

(ii)

$$B_n = 2I_n - I_{n-1} - I_{n+1}$$

where $I_m = \pm 1$. Hence :

$$\begin{array}{c} P(B_n = 0) = 1/4 \\ P(B_n = -2) = 1/4 \\ P(B_n = 2) = 1/4 \\ P(B_n = -4) = 1/8 \\ P(B_n = 4) = 1/8 \end{array}$$

Problem 9.11 :

The bandwidth of the bandpass channel is $W = 4$ KHz. Hence, the rate of transmission should be less or equal to 4000 symbols/sec. If a 8-QAM constellation is employed, then the required symbol rate is $R = 9600/3 = 3200$. If a signal pulse with raised cosine spectrum is used for shaping, the maximum allowable roll-off factor is determined by :

$$1600(1 + \beta) = 2000$$

which yields $\beta = 0.25$. Since β is less than 50%, we consider a larger constellation. With a 16-QAM constellation we obtain :

$$R = \frac{9600}{4} = 2400$$

and :

$$1200(1 + \beta) = 2000$$

or $\beta = 2/3$, which satisfies the required conditions. The probability of error for an M-QAM constellation is given by :

$$P_M = 1 - (1 - P_{\sqrt{M}})^2$$

where :

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

With $P_M = 10^{-6}$ we obtain $P_{\sqrt{M}} = 5 \times 10^{-7}$ and therefore using the last equation and the table of values for the $Q(\cdot)$ function, we find that the average transmitted energy is :

$$\mathcal{E}_{av} = 24.70 \times 10^{-9}$$

Note that if the desired spectral characteristic $X_{rc}(f)$ is split evenly between the transmitting and receiving filter, then the energy of the transmitting pulse is :

$$\int_{-\infty}^{\infty} g_T^2(t)dt = \int_{-\infty}^{\infty} |G_T(f)|^2 df = \int_{-\infty}^{\infty} X_{rc}(f)df = 1$$

Hence, the energy $\mathcal{E}_{av} = P_{av}T$ depends only on the amplitude of the transmitted points and the symbol interval T. Since $T = \frac{1}{2400}$, the average transmitted power is :

$$P_{av} = \frac{\mathcal{E}_{av}}{T} = 24.70 \times 10^{-9} \times 2400 = 592.8 \times 10^{-7}$$

If the points of the 16-QAM constellation are evenly spaced with minimum distance between them equal to d, then there are four points with coordinates $(\pm\frac{d}{2}, \pm\frac{d}{2})$, four points with coordinates $(\pm\frac{3d}{2}, \pm\frac{3d}{2})$, and eight points with coordinates $(\pm\frac{3d}{2}, \pm\frac{d}{2})$, or $(\pm\frac{d}{2}, \pm\frac{3d}{2})$. Thus, the average transmitted power is :

$$P_{av} = \frac{1}{2 \times 16}\sum_{i=1}^{16}(A_{mc}^2 + A_{ms}^2) = \frac{1}{32}\left[4 \times \frac{d^2}{2} + 4 \times \frac{9d^2}{2} + 8 \times \frac{10d^2}{4}\right] = \frac{5}{4}d^2$$

Since $P_{av} = 592.8 \times 10^{-7}$, we obtain

$$d = \sqrt{4\frac{P_{av}}{5}} = 0.0069$$

Problem 9.12 :

The channel (bandpass) bandwidth is $W = 4000$ Hz. Hence, the lowpass equivalent bandwidth will extend from -2 to 2 KHz.

(a) Binary PAM with a pulse shape that has $\beta = \frac{1}{2}$. Hence :

$$\frac{1}{2T}(1 + \beta) = 2000$$

so $\frac{1}{T} = 2667$, and since $k = 1$ bit/symbols is transmitted, the bit rate is 2667 bps.

(b) Four-phase PSK with a pulse shape that has $\beta = \frac{1}{2}$. From (a) the symbol rate is $\frac{1}{T} = 2667$ and the bit rate is 5334 bps.

(c) $M = 8$ QAM with a pulse shape that has $\beta = \frac{1}{2}$. From (a), the symbol rate is $\frac{1}{T} = 2667$ and hence the bit rate $\frac{3}{T} = 8001$ bps.

(d) Binary FSK with noncoherent detection. Assuming that the frequency separation between the two frequencies is $\Delta f = \frac{1}{T}$, where $\frac{1}{T}$ is the bit rate, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since $W = 4000$ Hz, we may select $\frac{1}{2T} = 1000$, or, equivalently, $\frac{1}{T} = 2000$. Hence, the bit rate is 2000 bps, and the two FSK signals are orthogonal.
(e) Four FSK with noncoherent detection. In this case we need four frequencies with separation of $\frac{1}{T}$ between adjacent frequencies. We select $f_1 = f_c - \frac{1.5}{T}$, $f_2 = f_c - \frac{1}{2T}$, $f_3 = f_c + \frac{1}{2T}$, and $f_4 = f_c + \frac{1.5}{T}$, where $\frac{1}{2T} = 500$ Hz. Hence, the symbol rate is $\frac{1}{T} = 1000$ symbols per second and since each symbol carries two bits of information, the bit rate is 2000 bps.
(f) $M = 8$ FSK with noncoherent detection. In this case we require eight frequencies with frequency separation of $\frac{1}{T} = 500$ Hz for orthogonality. Since each symbol carries 3 bits of information, the bit rate is 1500 bps.

Problem 9.13 :

(a) The bandwidth of the bandpass channel is :

$$W = 3000 - 600 = 2400 \text{ Hz}$$

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is :

$$R = \frac{1}{T} = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

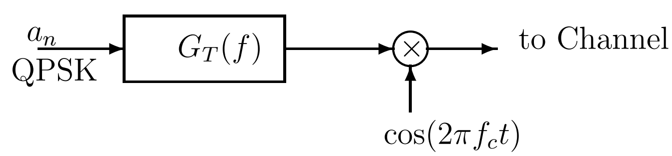
Thus, for spectral shaping we can use a signal pulse with a raised cosine spectrum and roll-off factor $\beta = 1$, since the spectral requirements will be $\frac{1}{2T}(1+\beta) = \frac{1}{T} = 1200$Hz. Hence :

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = \frac{1}{1200}\cos^2\left(\frac{\pi|f|}{2400}\right)$$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then

$$G_T(f) = G_R(f) = \sqrt{\frac{1}{1200}}\cos\left(\frac{\pi|f|}{2400}\right), \qquad |f| < \frac{1}{T} = 1200$$

A block diagram of the transmitter is shown in the next figure.



(b) If the bit rate is 4800 bps, then the symbol rate is

$$R = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, the the signal pulse used for spectral shaping, should have roll-off factor $\beta = 0$ with corresponding spectrum :

$$X(f) = T, \quad |f| < 1200$$

Thus, the frequency response of the transmitting filter is $G_T(f) = \sqrt{T}, \quad |f| < 1200$.

Problem 9.14 :

The bandwidth of the bandpass channel is :

$$W = 3300 - 300 = 3000 \text{ Hz}$$

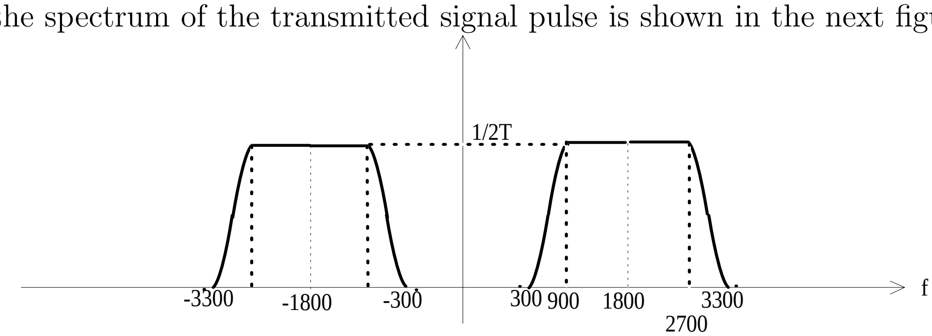
In order to transmit 9600 bps with a symbor rate $R = \frac{1}{T} = 2400$ symbols per second, the number of information bits per symbol should be :

$$k = \frac{9600}{2400} = 4$$

Hence, a $2^4 = 16$ QAM signal constellation is needed. The carrier frequency f_c is set to 1800 Hz, which is the mid-frequency of the frequency band that the bandpass channel occupies. If a pulse with raised cosine spectrum and roll-off factor β is used for spectral shaping, then for the bandpass signal with bandwidth W :

$$\frac{1}{2T}(1+\beta) = \frac{W}{2} = 1500 \Rightarrow \beta = 0.25$$

A sketch of the spectrum of the transmitted signal pulse is shown in the next figure.



Problem 9.15 :

The SNR at the detector is :

$$\frac{\mathcal{E}_b}{N_0} = \frac{P_b T}{N_0} = \frac{P_b(1+\beta)}{N_0 W} = 30 \text{ dB}$$

Since it is desired to expand the bandwidth by a factor of $\frac{10}{3}$ while maintaining the same SNR, the received power P_b should increase by the same factor. Thus the additional power needed is

$$P_a = 10\log_{10}\frac{10}{3} = 5.2288 \text{ dB}$$

Hence, the required transmitted power is :

$$P_S = -3 + 5.2288 = 2.2288 \text{ dBW}$$

Problem 9.16 :

The pulse $x(t)$ having the raised cosine spectrum given by (9-2-26/27) is :

$$x(t) = \text{sinc}(t/T)\frac{\cos(\pi\beta t/T)}{1-4\beta^2t^2/T^2}$$

The function sinc(t/T) is 1 when $t=0$ and 0 when $t=nT$. Therefore, the Nyquist criterion will be satisfied as long as the function $g(t)$ is :

$$g(t) = \frac{\cos(\pi\beta t/T)}{1-4\beta^2t^2/T^2} = \begin{cases} 1 & t=0 \\ \text{bounded} & t\neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\beta^2t^2/T^2 = 1$ or $\beta t = \frac{T}{2}$. However :

$$\lim_{\beta t\to\frac{T}{2}}\frac{\cos(\pi\beta t/T)}{1-4\beta^2t^2/T^2} = \lim_{x\to 1}\frac{\cos(\frac{\pi}{2}x)}{1-x}$$

and by using L'Hospital's rule :

$$\lim_{x\to 1}\frac{\cos(\frac{\pi}{2}x)}{1-x} = \lim_{x\to 1}\frac{\pi}{2}\sin(\frac{\pi}{2}x) = \frac{\pi}{2} < \infty$$

Hence :

$$x(nT) = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Problem 9.17 :

Substituting the expression of $X_{rc}(f)$ given by (8.2.22) in the desired integral, we obtain :

$$\int_{-\infty}^{\infty} X_{rc}(f)df = \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \frac{T}{2}\left[1+\cos\frac{\pi T}{\beta}(-f-\frac{1-\beta}{2T})\right]df + \int_{-\frac{1-\beta}{2T}}^{\frac{1-\beta}{2T}} Tdf$$

$$
\begin{aligned}
& + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2}\left[1+\cos\frac{\pi T}{\beta}(f-\frac{1-\beta}{2T})\right]df \\
= & \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \frac{T}{2}df + T\left(\frac{1-\beta}{T}\right) + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \frac{T}{2}df \\
& + \int_{-\frac{1+\beta}{2T}}^{-\frac{1-\beta}{2T}} \cos\frac{\pi T}{\beta}(f+\frac{1-\beta}{2T})df + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} \cos\frac{\pi T}{\beta}(f-\frac{1-\beta}{2T})df \\
= & \ 1 + \int_{-\frac{\beta}{T}}^{0} \cos\frac{\pi T}{\beta}xdx + \int_{0}^{\frac{\beta}{T}} \cos\frac{\pi T}{\beta}xdx \\
= & \ 1 + \int_{-\frac{\beta}{T}}^{\frac{\beta}{T}} \cos\frac{\pi T}{\beta}xdx = 1+0 = 1
\end{aligned}
$$

Problem 9.18 :

Let $X(f)$ be such that

$$
\mathrm{Re}[X(f)] = \begin{cases} T\Pi(fT)+U(f) & |f|<\frac{1}{T} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{Im}[X(f)] = \begin{cases} V(f) & |f|<\frac{1}{T} \\ 0 & \text{otherwise} \end{cases}
$$

with $U(f)$ even with respect to 0 and odd with respect to $f=\frac{1}{2T}$ Since $x(t)$ is real, $V(f)$ is odd with respect to 0 and by assumption it is even with respect to $f=\frac{1}{2T}$. Then,

$$
\begin{aligned}
x(t) &= \mathcal{F}^{-1}[X(f)] \\
&= \int_{-\frac{1}{T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{\frac{1}{2T}}^{\frac{1}{T}} X(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} Te^{j2\pi ft}df + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f)+jV(f)]e^{j2\pi ft}df \\
&= \mathrm{sinc}(t/T) + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f)+jV(f)]e^{j2\pi ft}df
\end{aligned}
$$

Consider first the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$. Clearly,

$$
\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df = \int_{-\frac{1}{T}}^{0} U(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} U(f)e^{j2\pi ft}df
$$

and by using the change of variables $f'=f+\frac{1}{2T}$ and $f'=f-\frac{1}{2T}$ for the two integrals on the right hand side respectively, we obtain

$$
\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df
$$

$$
\begin{aligned}
&= e^{-j\frac{\pi}{T}t}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'-\frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df' \\
&\stackrel{a}{=} \left(e^{j\frac{\pi}{T}t} - e^{-j\frac{\pi}{T}t}\right)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df' \\
&= 2j\sin(\frac{\pi}{T}t)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}
$$

where for step (a) we used the odd symmetry of $U(f')$ with respect to $f' = \frac{1}{2T}$, that is

$$
U(f'-\frac{1}{2T}) = -U(f'+\frac{1}{2T})
$$

For the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df$ we have

$$
\begin{aligned}
&\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{T}}^{0} V(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= e^{-j\frac{\pi}{T}t}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'-\frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t}\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'+\frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}
$$

However, $V(f)$ is odd with respect to 0 and since $V(f'+\frac{1}{2T})$ and $V(f'-\frac{1}{2T})$ are even, the translated spectra satisfy

$$
\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'-\frac{1}{2T})e^{j2\pi f't}df' = -\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f'+\frac{1}{2T})e^{j2\pi f't}df'
$$

Hence,

$$
\begin{aligned}
x(t) &= \text{sinc}(t/T) + 2j\sin(\frac{\pi}{T}t)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df' \\
&\quad -2\sin(\frac{\pi}{T}t)\int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f'+\frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}
$$

and therefore,

$$
x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}
$$

Thus, the signal $x(t)$ satisfies the Nyquist criterion.

Problem 9.19 :

The bandwidth of the channel is :

$$
W = 3000 - 300 = 2700 \text{ Hz}
$$

Since the minimum transmission bandwidth required for bandpass signaling is R, where R is the rate of transmission, we conclude that the maximum value of the symbol rate for the given channel is $R_{\max} = 2700$. If an M-ary PAM modulation is used for transmission, then in order to achieve a bit-rate of 9600 bps, with maximum rate of $R_{\max}$, the minimum size of the constellation is $M = 2^k = 16$. In this case, the symbol rate is :

$$R = \frac{9600}{k} = 2400 \text{ symbols/sec}$$

and the symbol interval $T = \frac{1}{R} = \frac{1}{2400}$ sec. The roll-off factor β of the raised cosine pulse used for transmission is is determined by noting that $1200(1+\beta) = 1350$, and hence, $\beta = 0.125$. Therefore, the squared root raised cosine pulse can have a roll-off of $\beta = 0.125$.

Problem 9.20 :

Since the one-sided bandwidth of the ideal lowpass channel is $W = 2400$ Hz, the rate of transmission is :

$$R = 2 \times 2400 = 4800 \text{ symbols/sec}$$

(remember that PAM can be transmitted single-sideband; hence, if the lowpass channel has bandwidth from -W to W, the passband channel will have bandwidth equal to W; on the other hand, a PSK or QAM system will have passband bandwidth equal to $2W$). The number of bits per symbol is

$$k = \frac{14400}{4800} = 3$$

Hence, the number of transmitted symbols is $2^3 = 8$. If a duobinary pulse is used for transmission, then the number of possible transmitted symbols is $2M - 1 = 15$. These symbols have the form

$$b_n = 0, \pm 2d, \pm 4d, \ldots, \pm 12d$$

where $2d$ is the minimum distance between the points of the 8-PAM constellation. The probability mass function of the received symbols is

$$P(b = 2md) = \frac{8 - |m|}{64}, \qquad m = 0, \pm 1, \ldots, \pm 7$$

An upper bound of the probability of error is given by (see (9-3-18))

$$P_M < 2\left(1 - \frac{1}{M^2}\right) Q\left[\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{M^2 - 1} \frac{k\mathcal{E}_{b,av}}{N_0}}\right]$$

With $P_M = 10^{-6}$ and $M = 8$ we obtain

$$\frac{k\mathcal{E}_{b,av}}{N_0} = 1.3193 \times 10^3 \Longrightarrow \mathcal{E}_{b,av} = 0.088$$

Problem 9.21 :

(a) The spectrum of the baseband signal is (see (4-4-12))

$$\Phi_V(f) = \frac{1}{T}\Phi_i i(f)|X_{rc}(f)|^2 = \frac{1}{T}|X_{rc}(f)|^2$$

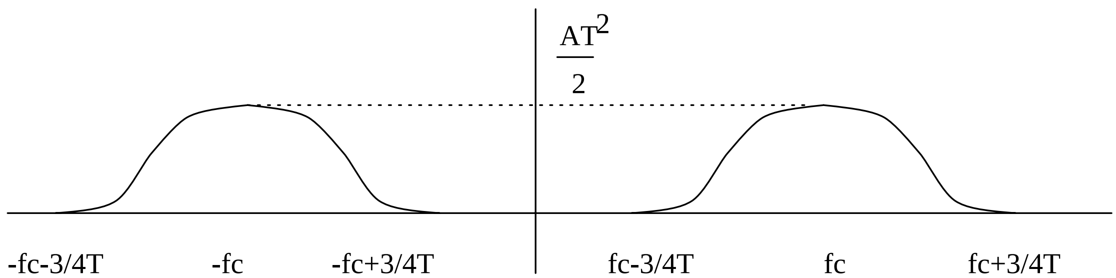
where $T = \frac{1}{2400}$ and

$$X_{rc}(f) = \begin{cases} T & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2}(1+\cos(2\pi T(|f|-\frac{1}{4T})) & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

If the carrier signal has the form $c(t) = A\cos(2\pi f_c t)$, then the spectrum of the DSB-SC modulated signal, $\Phi_U(f)$, is

$$\Phi_U(f) = \frac{A}{2}[\Phi_V(f-f_c) + \Phi_V(f+f_c)]$$

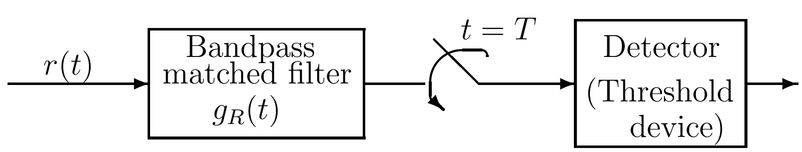
A sketch of $\Phi_U(f)$ is shown in the next figure.



(b) Assuming bandpass coherent demodulation using a matched filter, the received signal $r(t)$ is first passed through a linear filter with impulse response

$$g_R(t) = Ax_{rc}(T-t)\cos(2\pi f_c(T-t))$$

The output of the matched filter is sampled at $t = T$ and the samples are passed to the detector. The detector is a simple threshold device that decides if a binary 1 or 0 was transmitted depending on the sign of the input samples. The following figure shows a block diagram of the optimum bandpass coherent demodulator.



Problem 9.22 :

(a) The power spectral density of $X(t)$ is given by (see (4-4-12))

$$\Phi_x(f) = \frac{1}{T}\Phi_a(f)|G_T(f)|^2$$

The Fourier transform of $g(t)$ is

$$G_T(f) = \mathcal{F}[g(t)] = AT\frac{\sin \pi fT}{\pi fT}e^{-j\pi fT}$$

Hence,

$$|G_T(f)|^2 = (AT)^2\text{sinc}^2(fT)$$

and therefore,

$$\Phi_x(f) = A^2T\Phi_a(f)\text{sinc}^2(fT) = A^2T\text{sinc}^2(fT)$$

(b) If $g_1(t)$ is used instead of $g(t)$ and the symbol interval is T, then

$$\begin{aligned}\Phi_x(f) &= \frac{1}{T}\Phi_a(f)|G_{2T}(f)|^2 \\ &= \frac{1}{T}(A2T)^2\text{sinc}^2(f2T) = 4A^2T\text{sinc}^2(f2T)\end{aligned}$$

(c) If we precode the input sequence as $b_n = a_n + \alpha a_{n-3}$, then

$$\phi_b(m) = \begin{cases} 1+\alpha^2 & m = 0 \\ \alpha & m = \pm 3 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the power spectral density $\Phi_b(f)$ is

$$\Phi_b(f) = 1 + \alpha^2 + 2\alpha\cos(2\pi f3T)$$

To obtain a null at $f = \frac{1}{3T}$, the parameter α should be such that

$$1 + \alpha^2 + 2\alpha\cos(2\pi f3T)|_{f=\frac{1}{3}} = 0 \Longrightarrow \alpha = -1$$

(c) The answer to this question is no. This is because $\Phi_b(f)$ is an analytic function and unless it is identical to zero it can have at most a countable number of zeros. This property of the analytic functions is also referred as the theorem of isolated zeros.

Problem 9.23 :

The roll-off factor β is related to the bandwidth by the expression $\frac{1+\beta}{T} = 2W$, or equivalently $R(1+\beta) = 2W$. The following table shows the symbol rate for the various values of the excess bandwidth and for $W = 1500$ Hz.

β	.25	.33	.50	.67	.75	1.00
R	2400	2256	2000	1796	1714	1500

The above results were obtained with the assumption that double-sideband PAM is employed, so the available lowpass bandwidth will be from $-W = \frac{3000}{2}$ to W Hz. If single-sideband transmission is used, then the spectral efficiency is doubled, and the above symbol rates R are doubled.

Problem 9.24 :

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a duobinary transmitting filter.

Data seq. D_n:		1	0	0	1	0	1	1	0	0	1	0
Precoded seq. P_n:	0	1	1	1	0	0	1	0	0	0	1	1
Transmitted seq. I_n:	-1	1	1	1	-1	-1	1	-1	-1	-1	1	1
Received seq. B_n:		0	2	2	0	-2	0	0	-2	-2	0	2
Decoded seq. D_n:		1	0	0	1	0	1	1	0	0	1	0

Problem 9.25 :

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a modified duobinary transmitting filter.

Data seq. D_n:			1	0	0	1	0	1	1	0	0	1	0
Precoded seq. P_n:	0	0	1	0	1	1	1	0	0	0	0	1	0
Transmitted seq. I_n:	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1
Received seq. B_n:			2	0	0	2	0	-2	-2	0	0	2	0
Decoded seq. D_n:			1	0	0	1	0	1	1	0	0	1	0

Problem 9.26 :

Let $X(z)$ denote the $\mathcal{Z}$-transform of the sequence x_n, that is

$$X(z) = \sum_n x_n z^{-n}$$

Then the precoding operation can be described as

$$P(z) = \frac{D(z)}{X(z)} \quad \text{mod} - M$$

where $D(z)$ and $P(z)$ are the $\mathcal{Z}$-transforms of the data and precoded dequences respectively. For example, if $M = 2$ and $X(z) = 1 + z^{-1}$ (duobinary signaling), then

$$P(z) = \frac{D(z)}{1 + z^{-1}} \Longrightarrow P(z) = D(z) - z^{-1}P(z)$$

which in the time domain is written as

$$p_n = d_n - p_{n-1}$$

and the subtraction is mod-2.
However, the inverse filter $\frac{1}{X(z)}$ exists only if x_0, the first coefficient of $X(z)$ is relatively prime with M. If this is not the case, then the precoded symbols p_n cannot be determined uniquely from the data sequence d_n.
In the example given in the book, where $x_0 = 2$ we note that whatever the value of d_n (0 or 1), the value of ($2d_n$ mod 2) will be zero, hence this precoding scheme cannot work.

Problem 9.27 :

(a) The frequency response of the RC filter is

$$C(f) = \frac{\frac{1}{j2\pi RCf}}{R + \frac{1}{j2\pi RCf}} = \frac{1}{1 + j2\pi RCf}$$

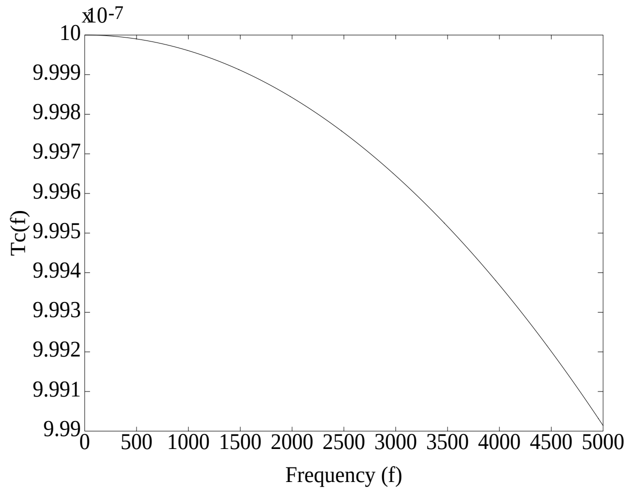
The amplitude and the phase spectrum of the filter are :

$$|C(f)| = \left(\frac{1}{1 + 4\pi^2(RC)^2 f^2}\right)^{\frac{1}{2}}, \qquad \theta_c(f) = \arctan(-2\pi RCf)$$

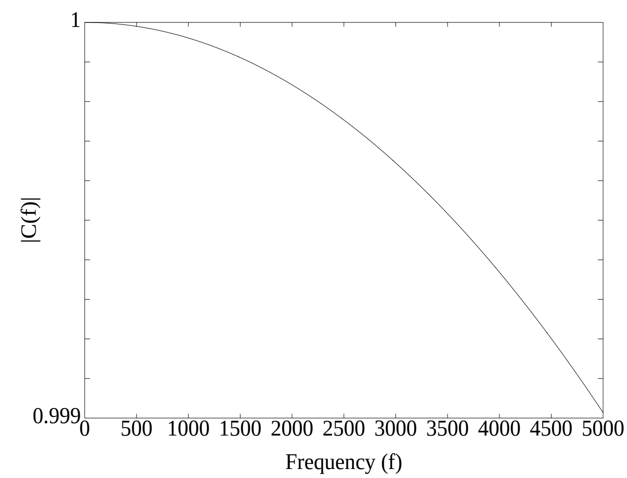
The envelope delay is

$$\tau_c(f) = -\frac{1}{2\pi}\frac{d\theta_c(f)}{df} = -\frac{1}{2\pi}\frac{-2\pi RC}{1 + 4\pi^2(RC)^2 f^2} = \frac{RC}{1 + 4\pi^2(RC)^2 f^2}$$

A plot of $\tau(f)$ with $RC = 10^{-6}$ is shown in the next figure :



(b) The following figure is a plot of the amplitude characteristics of the RC filter, $|C(f)|$. The values of the vertical axis indicate that $|C(f)|$ can be considered constant for frequencies up to 2000 Hz. Since the same is true for the envelope delay, we conclude that a lowpass signal of bandwidth $\Delta f = 1$ KHz will not be distorted if it passes the RC filter.



Problem 9.28 :

Let $G_T(f)$ and $G_R(f)$ be the frequency response of the transmitting and receiving filter. Then, the condition for zero ISI implies

$$G_T(f)C(f)G_R(f) = X_{rc}(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{1}{4T} \\ \frac{T}{2}[1+\cos(2\pi T(|f| - \frac{1}{T})], & \frac{1}{4T} \leq |f| \leq \frac{3}{4T} \\ 0, & |f| > \frac{3}{4T} \end{cases}$$

Since the additive noise is white, the optimum tansmitting and receiving filter characteristics are given by (see (9-2-81))

$$|G_T(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}, \qquad |G_R(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}$$

Thus,

$$|G_T(f)| = |G_R(f)| = \begin{cases} \left[\frac{T}{1+0.3\cos 2\pi fT}\right]^{\frac{1}{2}}, & 0 \le |f| \le \frac{1}{4T} \\ \left[\frac{T(1+\cos(2\pi T(|f|-\frac{1}{T})}{2(1+0.3\cos 2\pi fT)}\right]^{\frac{1}{2}}, & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0, & \text{otherwise} \end{cases}$$

Problem 9.29 :

A 4-PAM modulation can accomodate $k = 2$ bits per transmitted symbol. Thus, the symbol interval duration is :

$$T = \frac{k}{9600} = \frac{1}{4800} \text{ sec}$$

Since, the channel's bandwidth is $W = 2400 = \frac{1}{2T}$, in order to achieve the maximum rate of transmission, $R_{\max} = \frac{1}{2T}$, the spectrum of the signal pulse should be :

$$X(f) = T\Pi\left(\frac{f}{2W}\right)$$

Then, the magnitude frequency response of the optimum transmitting and receiving filter is (see (9-2-81))

$$|G_T(f)| = |G_R(f)| = \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}} \Pi\left(\frac{f}{2W}\right) = \begin{cases} \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}}, & |f| < 2400 \\ 0, & \text{otherwise} \end{cases}$$

Problem 9.30 :

We already know that

$$\sigma_v^2 = \int_{-\infty}^{\infty} \Phi_{nn}(f)|G_R(f)|^2 df$$

$$P_{av} = \frac{d^2}{T}\int_{-W}^{W} |G_T(f)|^2 df$$

$$|G_T(f)| = \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|}, |f| \le W$$

From these

$$\frac{\sigma_v^2}{d^2} = \frac{1}{P_{av}T}\int_{-W}^{W}\Phi_{nn}(f)|G_R(f)|^2 df \int_{-W}^{W}\frac{|X_{rc}(f)|^2}{|G_R(f)|^2|C(f)|^2}df \tag{4}$$

The optimum $|G_R(f)|$ can be found by applying the Cauchy-Schwartz inequality

$$\int_{-\infty}^{\infty}|U_1(f)|^2 df \int_{-\infty}^{\infty}|U_2(f)|^2 df \geq \left[\int_{-\infty}^{\infty}|U_1(f)||U_2(f)|df\right]^2$$

where $|U_1(f)|, |U_2(f)|$ are defined as

$$|U_1(f)| = |\sqrt{\Phi_{nn}(f)}||G_R(f)|$$

$$|U_2(f)| = \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|}$$

The minimum value of (1) is obtained when $|U_1(f)|$ is proportional to $|U_2(f)|$, i.e. $|U_1(f)| = K|U_2(f)|$ or, equivalently, when

$$|G_R(f)| = \sqrt{K}\frac{|X_{rc}(f)|^{1/2}}{[N_0/2]^{1/4}|C(f)|^{1/2}}, \quad |f| \leq W$$

where K is an arbitrary constant. By setting it appropriately,

$$|G_R(f)| = \frac{|X_{rc}(f)|^{1/2}}{|C(f)|^{1/2}}, \quad |f| \leq W$$

The corresponding modulation filter has a magnitude characteristic of

$$|G_T(f)| = \frac{|X_{rc}(f)|}{|G_R(f)||C(f)|} = \frac{|X_{rc}(f)|^{1/2}}{|C(f)|^{1/2}}, \quad |f| \leq W$$

Problem 9.31 :

In the case where the channel distortion is fully precompensated at the transmitter, the loss of SNR is given by

$$10\log L_1, \quad \text{with} \quad L_1 = \int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|^2}$$

whereas in the case of the equally split filters, the loss of SNR is given by

$$10\log[L_2]^2, \quad \text{with} \quad L_2 = \int_{-W}^{W}\frac{X_{rc}(f)}{|C(f)|}$$

Assuming that $1/T = W$, so that we have a raised cosine characteristic with $\beta = 0$, we have

$$X_{rc}(f) = \frac{1}{2W}\left[1+\cos\frac{\pi|f|}{W}\right]$$

Then

$$\begin{aligned} L_1 &= 2\int_0^W \frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{|C(f)|^2} \\ &= 2\left[\int_0^{W/2}\frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{1} + \int_{W/2}^{W}\frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{1/4}\right] \\ &= \frac{5\pi-6}{2\pi} \end{aligned}$$

Hence, the loss for the first type of filters is $10\log L_1 = 1.89$ dB.
In a similar way,

$$\begin{aligned} L_2 &= 2\int_0^W \frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{|C(f)|} \\ &= 2\left[\int_0^{W/2}\frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{1} + \int_{W/2}^{W}\frac{1}{2W}\frac{[1+\cos\frac{\pi f}{W}]}{1/2}\right] \\ &= \frac{3\pi-2}{2\pi} \end{aligned}$$

Hence, the loss for the second type of filters is $10\log[L_2]^2 = 1.45$ dB. As expected, the second type of filters which split the channel characteristics between the transmitter and the receiver exhibit a smaller SNR loss.

Problem 9.32 :

The state transition matrix of the (0,1) runlength-limited code is :

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of D are the roots of

$$\det(D-\lambda I) = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1$$

The roots of the characteristic equation are :

$$\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$$

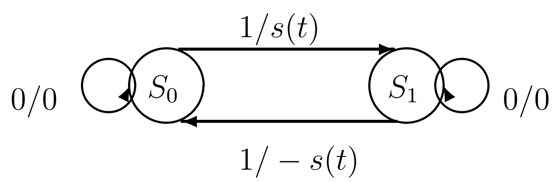
Thus, the capacity of the (0,1) runlength-limited code is :

$$C(0,1) = \log_2(\frac{1\pm\sqrt{5}}{2}) = 0.6942$$

The capacity of a $(1,\infty)$ code is found from Table 9-4-1 to be 0.6942. As it is observed, the two codes have exactly the same capacity. This result is to be expected since the (0,1) runlength-limited code and the $(1,\infty)$ code produce the same set of code sequences of length n, $N(n)$, with a renaming of the bits from 0 to 1 and vise versa. For example, the (0,1) runlength-limited code with a renaming of the bits, can be described as the code with no minimum number of 1's between 0's in a sequence, and at most one 1 between two 0's. In terms of 0's, this is simply the code with no restrictions on the number of adjacent 0's and no consequtive 1's, that is the $(1, \infty)$ code.

Problem 9.33 :

Let S_0 represent the state that the running polarity is zero, and S_1 the state that there exists some polarity (dc component). The following figure depicts the transition state diagram of the AMI code :



The state transition matrix is :

$$D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues of the matrix D can be found from

$$\det(D - \lambda I) = 0 \Longrightarrow (1-\lambda)^2 - 1 = 0 \text{ or } \lambda(2-\lambda) = 0$$

The largest real eigenvalue is $\lambda_{\max} = 2$, so that :

$$C = \log_2 \lambda_{\max} = 1$$

Problem 9.34 :

Let $\{b_k\}$ be a binary sequence, taking the values 1, 0 depending on the existence of polarization at the transmitted sequence up to the time instant k. For the AMI code, b_k is expressed as

$$b_k = a_k \oplus b_{k-1} = a_k \oplus a_{k-1} \oplus a_{k-2} \oplus \ldots$$

where $\oplus$ denotes modulo two addition. Thus, the AMI code can be described as the RDS code, with RDS ($=b_k$) denoting the binary digital sum modulo 2 of the input bits.

Problem 9.35 :

Defining the efficiency as :

$$\text{efficiency} = \frac{k}{n \log_2 3}$$

we obtain :

Code	Efficiency
1B1T	0.633
3B2T	0.949
4B3T	0.844
6B4T	0.949

Problem 9.36 :

(a) The characteristic polynomial of D is :

$$\det(D - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1$$

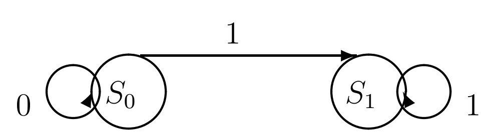
The eigenvalues of D are the roots of the characteristic polynomial, that is

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the largest eigenvalue of D is $\lambda_{\max} = \frac{1+\sqrt{5}}{2}$ and therefore :

$$C = \log_2 \frac{1+\sqrt{5}}{2} = 0.6942$$

(b) The characteristic polynomial is $\det(D - \lambda I) = (1-\lambda)^2$ with roots $\lambda_{1,2} = 1$. Hence, $C = \log_2 1 = 0$. The state diagram of this code is depicted in the next figure.



(c) As it is observed the second code has zero capacity. This result is to be expected since with the second code we can have at most $n+1$ different sequences of length n, so that

$$C = \lim_{n\to\infty} \frac{1}{n} \log_2 N(n) = \lim_{n\to\infty} \frac{1}{n} \log_2 (n+1) = 0$$

The $n+1$ possible sequences are

$$\underbrace{0\ldots0}_{k}\underbrace{1\ldots1}_{n-k} \qquad (n \text{ sequences})$$

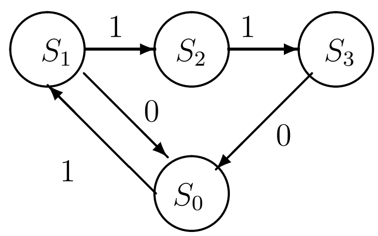
and the sequence $11\ldots1$, which occurs if we start from state S_1.

Problem 9.37 :

(a) The two symbols, dot and dash, can be represented as 10 and 1110 respectively, where 1 denotes line closure and 0 an open line. Hence, the constraints of the code are :

- A 0 is always followed by 1.
- Only sequences having one or three repetitions of 1, are allowed.

The next figure depicts the state diagram of the code, where the state S_0 denotes the reception of a dot or a dash, and state S_i denotes the reception of i adjacent 1's.



(b) The state transition matrix is :

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(c) The characteristic equation of the matrix D is :

$$\det(D - \lambda I) = 0 \Longrightarrow \lambda^4 - \lambda^2 - 1 = 0$$

The roots of the characteristic equation are :

$$\lambda_{1,2} = \pm \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{1}{2}} \qquad \lambda_{3,4} = \pm \left(\frac{1-\sqrt{5}}{2}\right)^{\frac{1}{2}}$$

Thus, the capacity of the code is :

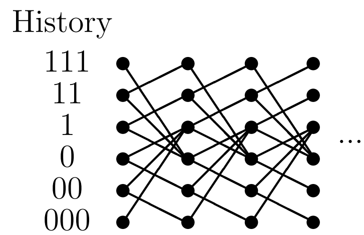
$$C = \log_2 \lambda_{\max} = \log_2 \lambda_1 = \log_2 \left(\frac{1+\sqrt{5}}{2} \right)^{\frac{1}{2}} = 0.3471$$

Problem 9.38 :

The state diagram of Fig. P9-31 describes a runlength constrained code, that forbids any sequence containing a run of more than three adjacent symbols of the same kind. The state transition matrix is :

$$D = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The corresponding trellis is shown in the next figure :



Problem 9.39 :

The state transition matrix of the (2,7) runlength-limited code is the 8×8 matrix :

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CHAPTER 10

Problem 10.1 :

Suppose that $a_m = +1$ is the transmitted signal. Then the probability of error will be :

$$\begin{aligned} P_{e|1} &= P\left(y_m < 0 | a_m = +1\right) \\ &= P\left(1 + n_m + i_m < 0\right) \\ &= \frac{1}{4}P\left(1/2 + n_m < 0\right) + \frac{1}{4}P\left(3/2 + n_m < 0\right) + \frac{1}{2}P\left(1 + n_m < 0\right) \\ &= \frac{1}{4}Q\left[\frac{1}{2\sigma_n}\right] + \frac{1}{4}Q\left[\frac{3}{2\sigma_n}\right] + \frac{1}{2}Q\left[\frac{1}{\sigma_n}\right] \end{aligned}$$

Due to the symmetry of the intersymbol interference, the probability of error, when $a_m = -1$ is transmitted, is the same. Thus, the above result is the average probability of error.

Problem 10.2 :

(a) If the transmitted signal is :

$$r(t) = \sum_{n=-\infty}^{\infty} I_n h(t - nT) + n(t)$$

then the output of the receiving filter is :

$$y(t) = \sum_{n=-\infty}^{\infty} I_n x(t - nT) + \nu(t)$$

where $x(t) = h(t) \star h(t)$ and $\nu(t) = n(t) \star h(t)$. If the sampling time is off by 10%, then the samples at the output of the correlator are taken at $t = (m \pm \frac{1}{10})T$. Assuming that $t = (m - \frac{1}{10})T$ without loss of generality, then the sampled sequence is :

$$y_m = \sum_{n=-\infty}^{\infty} I_n x((m - \frac{1}{10}T - nT) + \nu((m - \frac{1}{10})T)$$

If the signal pulse is rectangular with amplitude A and duration T, then $\sum_{n=-\infty}^{\infty} I_n x((m - \frac{1}{10}T - nT)$ is nonzero only for $n = m$ and $n = m - 1$ and therefore, the sampled sequence is given by :

$$\begin{aligned} y_m &= I_m x(-\frac{1}{10}T) + I_{m-1} x(T - \frac{1}{10}T) + \nu((m - \frac{1}{10})T) \\ &= \frac{9}{10} I_m A^2 T + I_{m-1}\frac{1}{10}A^2 T + \nu((m - \frac{1}{10})T) \end{aligned}$$

The variance of the noise is :

$$\sigma_\nu^2 = \frac{N_0}{2} A^2 T$$

and therefore, the SNR is :

$$\text{SNR} = \left(\frac{9}{10}\right)^2 \frac{2(A^2T)^2}{N_0 A^2 T} = \frac{81}{100}\frac{2A^2T}{N_0}$$

As it is observed, there is a loss of $10\log_{10}\frac{81}{100} = -0.9151$ dB due to the mistiming.

(b) Recall from part (a) that the sampled sequence is

$$y_m = \frac{9}{10} I_m A^2 T + I_{m-1}\frac{1}{10}A^2T + \nu_m$$

The term $I_{m-1}\frac{A^2T}{10}$ expresses the ISI introduced to the system. If $I_m = 1$ is transmitted, then the probability of error is

$$\begin{aligned}
P(e|I_m = 1) &= \frac{1}{2}P(e|I_m = 1, I_{m-1} = 1) + \frac{1}{2}P(e|I_m = 1, I_{m-1} = -1) \\
&= \frac{1}{2\sqrt{\pi N_0 A^2 T}}\int_{-\infty}^{-A^2T} e^{-\frac{\nu^2}{N_0 A^2 T}}d\nu + \frac{1}{2\sqrt{\pi N_0 A^2 T}}\int_{-\infty}^{-\frac{8}{10}A^2T} e^{-\frac{\nu^2}{N_0 A^2 T}}d\nu \\
&= \frac{1}{2}Q\left[\sqrt{\frac{2A^2T}{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\left(\frac{8}{10}\right)^2\frac{2A^2T}{N_0}}\right]
\end{aligned}$$

Since the symbols of the binary PAM system are equiprobable the previous derived expression is the probability of error when a symbol by symbol detector is employed. Comparing this with the probability of error of a system with no ISI, we observe that there is an increase of the probability of error by

$$P_{\text{diff}}(e) = \frac{1}{2}Q\left[\sqrt{\left(\frac{8}{10}\right)^2\frac{2A^2T}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 10.3 :

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \frac{\alpha}{2}\delta(t - t_0) + \frac{\alpha}{2}\delta(t + t_0)$$

Hence,

$$y(t) = s(t) \star h(t) = s(t) + \frac{\alpha}{2}s(t - t_0) + \frac{\alpha}{2}s(t + t_0)$$

(b) If the signal $s(t)$ is used to modulate the sequence $\{I_n\}$, then the transmitted signal is :

$$u(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT)$$

The received signal is the convolution of $u(t)$ with $h(t)$. Hence,

$$\begin{aligned} y(t) &= u(t) \star h(t) = \left(\sum_{n=-\infty}^{\infty} I_n s(t - nT) \right) \star \left(\delta(t) + \frac{\alpha}{2}\delta(t - t_0) + \frac{\alpha}{2}\delta(t + t_0) \right) \\ &= \sum_{n=-\infty}^{\infty} I_n s(t - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n s(t - t_0 - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n s(t + t_0 - nT) \end{aligned}$$

Thus, the output of the matched filter $s(-t)$ at the time instant t_1 is :

$$\begin{aligned} w(t_1) &= \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau - nT) s(\tau - t_1) d\tau \\ &+ \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau - t_0 - nT) s(\tau - t_1) d\tau \\ &+ \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n \int_{-\infty}^{\infty} s(\tau + t_0 - nT) s(\tau - t_1) d\tau \end{aligned}$$

If we denote the signal $s(t) \star s(t)$ by $x(t)$, then the output of the matched filter at $t_1 = kT$ is :

$$\begin{aligned} w(kT) &= \sum_{n=-\infty}^{\infty} I_n x(kT - nT) \\ &+ \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n x(kT - t_0 - nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} I_n x(kT + t_0 - nT) \end{aligned}$$

(c) With $t_0 = T$ and $k = n$ in the previous equation, we obtain :

$$\begin{aligned} w_k &= I_k x_0 + \sum_{n \neq k} I_n x_{k-n} \\ &+ \frac{\alpha}{2} I_k x_{-1} + \frac{\alpha}{2} \sum_{n \neq k} I_n x_{k-n-1} + \frac{\alpha}{2} I_k x_1 + \frac{\alpha}{2} \sum_{n \neq k} I_n x_{k-n+1} \\ &= I_k \left(x_0 + \frac{\alpha}{2} x_{-1} + \frac{\alpha}{2} x_1 \right) + \sum_{n \neq k} I_n \left[x_{k-n} + \frac{\alpha}{2} x_{k-n-1} + \frac{\alpha}{2} x_{k-n+1} \right] \end{aligned}$$

The terms under the summation is the ISI introduced by the channel. If the signal $s(t)$ is designed so as to satisfy the Nyquist criterion, then :

$$x_k = 0, \; k \neq 0$$

and the aobove expression simplifies to :

$$w_k = I_k + \frac{\alpha}{2}(I_{k+1} + I_{k-1})$$

Problem 10.4 :

(a) Each segment of the wire-line can be considered as a bandpass filter with bandwidth $W = 1200$ Hz. Thus, the highest bit rate that can be transmitted without ISI by means of binary PAM is :

$$R = 2W = 2400 \text{ bps}$$

(b) The probability of error for binary PAM trasmission is :

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Hence, using mathematical tables for the function $Q[\cdot]$, we find that $P_2 = 10^{-7}$ is obtained for :

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.2 \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 13.52 = 11.30 \text{ dB}$$

(c) The received power P_R is related to the desired SNR per bit through the relation :

$$\frac{P_R}{N_0} = \frac{1}{T}\frac{\mathcal{E}_b}{N_0} = R\frac{\mathcal{E}_b}{N_0}$$

Hence, with $N_0 = 4.1 \times 10^{-21}$ we obtain :

$$P_R = 4.1 \times 10^{-21} \times 1200 \times 13.52 = 6.6518 \times 10^{-17} = -161.77 \text{ dBW}$$

Since the power loss of each segment is :

$$L_s = 50 \text{ Km } \times 1 \text{ dB/Km } = 50 \text{ dB}$$

the transmitted power at each repeater should be :

$$P_T = P_R + L_s = -161.77 + 50 = -111.77 \text{ dBW}$$

Problem 10.5 :

$$x_n = \int_{-\infty}^{\infty} h(t + nT)h^*(t)dt$$

$$v_k = \int_{-\infty}^{\infty} z(t)h^*(t-kT)dt$$

Then :

$$\begin{aligned}
\frac{1}{2}E\left[v_j v_k^*\right] &= \frac{1}{2}E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z(a)h^*(a-jT)z^*(b)h(b-kT)dadb\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}E\left[z(a)z^*(b)\right] h^*(a-jT)h(b-kT)dadb \\
&= N_0 \int_{-\infty}^{\infty} h^*(a-jT)h(a-kT)da = N_0 x_{j-k}
\end{aligned}$$

Problem 10.6 :

In the case of duobinary signaling, the output of the matched filter is :

$$x(t) = \text{sinc}(2Wt) + \text{sinc}(2Wt-1)$$

and the samples x_{n-m} are given by :

$$x_{n-m} = x(nT-mT) = \begin{cases} 1 & n-m=0 \\ 1 & n-m=1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the metric $CM(\mathbf{I})$ in the Viterbi algorithm becomes

$$\begin{aligned}
CM(\mathbf{I}) &= 2\sum_n I_n r_n - \sum_n\sum_m I_n I_m x_{n-m} \\
&= 2\sum_n I_n r_n - \sum_n I_n^2 - \sum_n I_n I_{n-1} \\
&= \sum_n I_n(2r_n - I_n - I_{n-1})
\end{aligned}$$

Problem 10.7 :

(a) The output of the matched filter demodulator is :

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} I_k \int_{-\infty}^{\infty} g_T(\tau - kT_b)g_R(t-\tau)d\tau + \nu(t) \\
&= \sum_{k=-\infty}^{\infty} I_k x(t-kT_b) + \nu(t)
\end{aligned}$$

where,

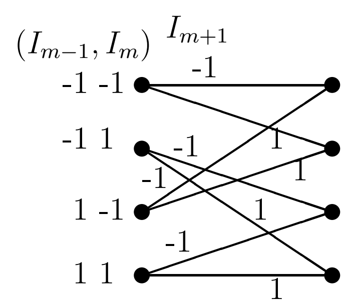
$$x(t) = g_T(t) \star g_R(t) = \frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}} \frac{\cos\frac{\pi t}{T}}{1-4\frac{t^2}{T^2}}$$

Hence,

$$\begin{aligned} y(mT_b) &= \sum_{k=-\infty}^{\infty} I_k x(mT_b - kT_b) + v(mT_b) \\ &= I_m + \frac{1}{\pi} I_{m-1} + \frac{1}{\pi} I_{m+1} + \nu(mT_b) \end{aligned}$$

The term $\frac{1}{\pi} I_{m-1} + \frac{1}{\pi} I_{m+1}$ represents the ISI introduced by doubling the symbol rate of transmission.

(b) In the next figure we show one trellis stage for the ML sequence detector. Since there is postcursor ISI, we delay the received signal, used by the ML decoder to form the metrics, by one sample. Thus, the states of the trellis correspond to the sequence (I_{m-1}, I_m), and the transition labels correspond to the symbol I_{m+1}. Two branches originate from each state. The upper branch is associated with the transmission of -1, whereas the lower branch is associated with the transmission of 1.



Problem 10.8 :

(a) The output of the matched filter at the time instant mT is :

$$y_m = \sum_k I_m x_{k-m} + \nu_m = I_m + \frac{1}{4} I_{m-1} + \nu_m$$

The autocorrelation function of the noise samples ν_m is

$$E[\nu_k \nu_j] = \frac{N_0}{2} x_{k-j}$$

Thus, the variance of the noise is

$$\sigma_\nu^2 = \frac{N_0}{2} x_0 = \frac{N_0}{2}$$

If a symbol by symbol detector is employed and we assume that the symbols $I_m = I_{m-1} = \sqrt{\mathcal{E}_b}$ have been transmitted, then the probability of error $P(e|I_m = I_{m-1} = \sqrt{\mathcal{E}_b})$ is :

$$P(e|I_m = I_{m-1} = \sqrt{\mathcal{E}_b}) \;\; = \;\; P(y_m < 0 | I_m = I_{m-1} = \sqrt{\mathcal{E}_b})$$

$$= \quad P(\nu_m < -\frac{5}{4}\sqrt{\mathcal{E}_b}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\mathcal{E}_b}} e^{-\frac{\nu_m^2}{N_0}} d\nu_m$$

$$= \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}} e^{-\frac{\nu^2}{2}} d\nu = Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

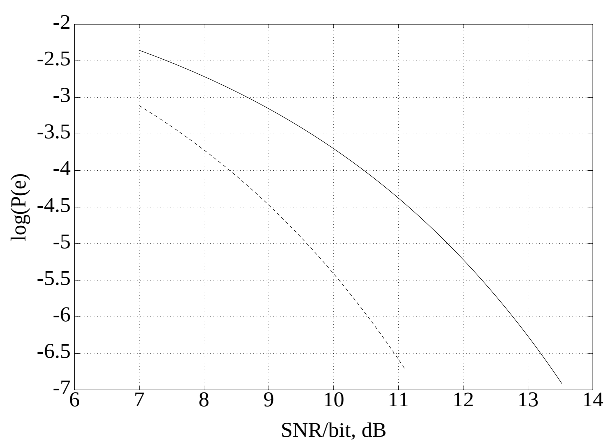
If however $I_{m-1} = -\sqrt{\mathcal{E}_b}$, then :

$$P(e|I_m = \sqrt{\mathcal{E}_b}, I_{m-1} = -\sqrt{\mathcal{E}_b}) = P(\frac{3}{4}\sqrt{\mathcal{E}_b} + \nu_m < 0) = Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Since the two symbols $\sqrt{\mathcal{E}_b}$, $-\sqrt{\mathcal{E}_b}$ are used with equal probability, we conclude that :

$$\begin{aligned} P(e) &= P(e|I_m = \sqrt{\mathcal{E}_b}) = P(e|I_m = -\sqrt{\mathcal{E}_b}) \\ &= \frac{1}{2}Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{2}Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \end{aligned}$$

(b) In the next figure we plot the error probability obtained in part (a) ($\log_{10}(P(e))$) vs. the SNR per bit and the error probability for the case of no ISI. As it observed from the figure, the relative difference in SNR of the error probability of 10^{-6} is 2 dB.



Problem 10.9 :

For the DFE we have that :

$$\hat{I}_k = \sum_{j=-K_1}^{0} c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}$$

We want to minimize $J = E\left|I_k - \hat{I}_k\right|^2$. Taking the derivative of J, with respect to the real and imaginary parts of $c_l = a_l + jb_l,\ 1 \leq l \leq K_2$, we obtain :

$$\begin{array}{c} \frac{\partial J}{\partial a_l} = 0 \Rightarrow E\left[-I_{k-l}\left(I_k^* - \hat{I}_k^*\right) - I_{k-l}^*\left(I_k - \hat{I}_k\right)\right] = 0 \Rightarrow \\ E\left[Re\left\{I_{k-l}^*\left(I_k - \hat{I}_k\right)\right\}\right] = 0 \end{array}$$

and similarly :

$$\frac{\partial J}{\partial b_l} = 0 \Rightarrow E\left[Im\left\{I_{k-l}^*\left(I_k - \hat{I}_k\right)\right\}\right] = 0$$

Hence,

$$E\left[I_{k-l}^*\left(I_k - \hat{I}_k\right)\right] = 0, \quad 1 \leq l \leq K_2 \qquad (1)$$

Since the information symbols are uncorrelated : $E\left[I_k I_l^*\right] = \delta_{kl}$. We also have :

$$\begin{array}{rcl} E\left[I_k u_l^*\right] & = & E\left[I_k\left(\sum_{m=0}^{L} f_m^* I_{l-m}^* + n_l^*\right)\right] \\ & = & f_{l-k}^* \end{array}$$

Hence, equation (1) gives :

$$\begin{array}{l} E\left[I_k I_{k-l}^*\right] = E\left[\hat{I}_k I_{k-l}^*\right], \quad 1 \leq l \leq K_2 \Rightarrow \\ 0 = E\left[\left(\sum_{j=-K_1}^{0} c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}\right) I_{k-l}^*\right] \Rightarrow \\ 0 = \left(\sum_{j=-K_1}^{0} c_j f_{l-j}\right) + c_l \Rightarrow \\ c_l = -\sum_{j=-K_1}^{0} c_j f_{l-j},\ 1 \leq l \leq K_2 \end{array}$$

which is the desired equation for the feedback taps.

Problem 10.10 :

(a) The equivalent discrete-time impulse response of the channel is :

$$h(t) = \sum_{n=-1}^{1} h_n \delta(t - nT) = 0.3\delta(t+T) + 0.9\delta(t) + 0.3\delta(t-T)$$

If by $\{c_n\}$ we denote the coefficients of the FIR equalizer, then the equalized signal is :

$$q_m = \sum_{n=-1}^{1} c_n h_{m-n}$$

which in matrix notation is written as :

$$\left(\begin{array}{ccc} 0.9 & 0.3 & 0. \\ 0.3 & 0.9 & 0.3 \\ 0. & 0.3 & 0.9 \end{array}\right)\left(\begin{array}{c} c_{-1} \\ c_0 \\ c_1 \end{array}\right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right)$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation. Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

(b) The values of q_m for $m = \pm 2, \pm 3$ are given by

$$\begin{aligned} q_2 &= \sum_{n=-1}^{1} c_n h_{2-n} = c_1 h_1 = -0.1429 \\ q_{-2} &= \sum_{n=-1}^{1} c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429 \\ q_3 &= \sum_{n=-1}^{1} c_n h_{3-n} = 0 \\ q_{-3} &= \sum_{n=-1}^{1} c_n h_{-3-n} = 0 \end{aligned}$$

Problem 10.11 :

(a) The output of the zero-force equalizer is :

$$q_m = \sum_{n=-1}^{1} c_n x_{m_n}$$

With $q_0 = 1$ and $q_m = 0$ for $m \neq 0$, we obtain the system :

$$\begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain :

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0.000 \\ 0.980 \\ 0.196 \end{pmatrix}$$

(b) The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -4 \\ c_{-1}x_{-2} = 0 & m = -3 \\ c_{-1}x_{-1} + c_0 x_{-2} = -0.49 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_0 x_2 + x_1 c_1 = 0.0098 & m = 2 \\ c_1 x_2 = 0.0098 & m = 3 \\ 0 & m \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \ldots\}$$

and its span is 6 symbols.

Problem 10.12 :

(a) If $\{c_n\}$ denote the coefficients of the zero-force equalizer and $\{q_m\}$ is the sequence of the equalizer's output samples, then :

$$q_m = \sum_{n=-1}^{1} c_n x_{m-n}$$

where $\{x_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_b$, we obtain the system :

$$\begin{pmatrix} \mathcal{E}_b & 0.9\mathcal{E}_b & 0.1\mathcal{E}_b \\ 0.9\mathcal{E}_b & \mathcal{E}_b & 0.9\mathcal{E}_b \\ 0.1\mathcal{E}_b & 0.9\mathcal{E}_b & \mathcal{E}_b \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}_b \\ \mathcal{E}_b \end{pmatrix}$$

The solution to the system is :

$$\begin{pmatrix} c_{-1} & c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0.2137 & -0.3846 & 1.3248 \end{pmatrix}$$

(b) The set of noise variables $\{\nu_k\}$ at the output of the sampler is a gaussian distributed sequence with zero-mean and autocorrelation function :

$$R_\nu(k) = \begin{cases} \frac{N_0}{2} x_k & |k| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the autocorrelation function of the noise at the output of the equalizer is :

$$R_n(k) = R_\nu(k) \star c(k) \star c(-k)$$

where $c(k)$ denotes the discrete time impulse response of the equalizer. Therefore, the autocorrelation sequence of the noise at the output of the equalizer is :

$$R_n(k) = \frac{N_0\mathcal{E}_b}{2}\begin{cases} 0.9402 & k=0 \\ 1.3577 & k=\pm 1 \\ -0.0546 & k=\pm 2 \\ 0.1956 & k=\pm 3 \\ 0.0283 & k=\pm 4 \\ 0 & \text{otherwise} \end{cases}$$

To find an estimate of the error probability for the sequence detector, we ignore the residual interference due to the finite length of the equalizer, and we only consider paths of length two. Thus, if we start at state $I_0 = 1$ and the transmitted symbols are $(I_1, I_2) = (1, 1)$ an error is made by the sequence detector if the path $(-1, 1)$ is more probable, given the received values of r_1 and r_2. The metric for the path $(I_1, I_2) = (1, 1)$ is :

$$\mu_2(1,1) = [\ r_1 - 2\mathcal{E}_b \quad r_2 - 2\mathcal{E}_b\ ]\mathbf{C}^{-1}\begin{bmatrix} r_1 - 2\mathcal{E}_b \\ r_2 - 2\mathcal{E}_b \end{bmatrix}$$

where :

$$\mathbf{C} = \frac{N_0\mathcal{E}_b}{2}\begin{pmatrix} 0.9402 & 1.3577 \\ 1.3577 & 0.9402 \end{pmatrix}$$

Similarly, the metric of the path $(I_1, I_2) = (-1, 1)$ is

$$\mu_2(-1,1) = [\ r_1 \quad r_2\ ]\mathbf{C}^{-1}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

Hence, the probability of error is :

$$P_2 = P(\mu_2(-1,1) < \mu_2(1,1))$$

and upon substitution of $r_1 = 2\mathcal{E}_b + n_1$, $r_2 = 2\mathcal{E}_b + n_2$, we obtain :

$$P_2 = P(n_1 + n_2 < -2\mathcal{E}_b)$$

Since n_1 and n_2 are zero-mean Gaussian variables, their sum is also zero-mean Gaussian with variance :

$$\sigma_2 = (2 \times 0.9402 + 2 \times 1.3577)\frac{N_0\mathcal{E}_b}{2} = 4.5958\frac{N_0\mathcal{E}_b}{2}$$

and therefore :

$$P_2 = Q\left[\sqrt{\frac{8\mathcal{E}_b}{4.5958 N_0}}\right]$$

The bit error probability is $\frac{P_2}{2}$.

Problem 10.13 :

The optimum tap coefficients of the zero-force equalizer can be found by solving the system:

$$\begin{pmatrix} 1.0 & 0.3 & 0.0 \\ 0.2 & 1.0 & 0.3 \\ 0.0 & 0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.3409 \\ 1.1364 \\ -0.2273 \end{pmatrix}$$

The output of the equalizer is :

$$q_m = \begin{cases} 0 & m \leq -3 \\ c_{-1}x_{-1} = -0.1023 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_1 x_1 = -0.0455 & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Hence, the residual ISI sequence is :

$$\text{residual ISI} = \{\ldots, 0, -0.1023, 0, 0, 0, -0.0455, 0, \ldots\}$$

Problem 10.14 :

(a) If we assume that the signal pulse has duration T, then the ouput of the matched filter at the time instant $t = T$ is :

$$\begin{aligned} y(T) &= \int_0^T r(\tau)s(\tau)d\tau \\ &= \int_0^T (s(\tau) + \alpha s(\tau - T) + n(\tau))s(\tau)d\tau \\ &= \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \mathcal{E}_s + n \end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal pulse and n is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{N_0 \mathcal{E}_s}{2}$. Similarly, the output of the matched filter at $t = 2T$ is :

$$\begin{aligned} y(2T) &= \alpha \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \alpha \mathcal{E}_s + n \end{aligned}$$

(b) If the transmitted sequence is :

$$x(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT)$$

with I_n taking the values $1, -1$ with equal probability, then the output of the demodulator at the time instant $t = kT$ is

$$y_k = I_k \mathcal{E}_s + \alpha I_{k-1} \mathcal{E}_s + n_k$$

The term $\alpha I_{k-1} \mathcal{E}_s$ expresses the ISI due to the signal reflection. If a symbol by symbol detector is employed and the ISI is ignored, then the probability of error is :

$$\begin{aligned} P(e) &= \frac{1}{2} P(\text{error}|I_n = 1, I_{n-1} = 1) + \frac{1}{2} P(\text{error}|I_n = 1, I_{n-1} = -1) \\ &= \frac{1}{2} P((1+\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2} P((1-\alpha)\mathcal{E}_s + n_k < 0) \\ &= \frac{1}{2} Q\left[\sqrt{\frac{2(1+\alpha)^2 \mathcal{E}_s}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{2(1-\alpha)^2 \mathcal{E}_s}{N_0}}\right] \end{aligned}$$

(c) To find the error rate performance of the DFE, we assume that the estimation of the parameter α is correct and that the probability of error at each time instant is the same. Since the transmitted symbols are equiprobable, we obtain :

$$\begin{aligned} P(e) &= P(\text{error at } k | I_k = 1) \\ &= P(\text{error at } k-1) P(\text{error at } k | I_k = 1, \text{error at } k-1) \\ &\quad + P(\text{no error at } k-1) P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\ &= P(e) P(\text{error at } k | I_k = 1, \text{error at } k-1) \\ &\quad + (1 - P(e)) P(\text{error at } k | I_k = 1, \text{no error at } k-1) \\ &= P(e) p + (1 - P(e)) q \end{aligned}$$

where :

$$\begin{aligned} p &= P(\text{error at } k | I_k = 1, \text{error at } k-1) \\ &= \frac{1}{2} P(\text{error at } k | I_k = 1, I_{k-1} = 1, \text{error at } k-1) \\ &\quad + \frac{1}{2} P(\text{error at } k | I_k = 1, I_{k-1} = -1, \text{error at } k-1) \\ &= \frac{1}{2} P((1+2\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2} P((1-2\alpha)\mathcal{E}_s + n_k < 0) \\ &= \frac{1}{2} Q\left[\sqrt{\frac{2(1+2\alpha)^2 \mathcal{E}_s}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\frac{2(1-2\alpha)^2 \mathcal{E}_s}{N_0}}\right] \end{aligned}$$

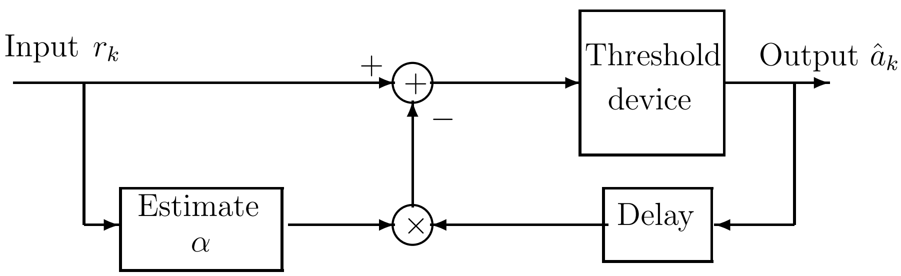
and

$$q = P(\text{error at } k|I_k = 1, \text{no error at } k-1)$$
$$= P(\mathcal{E}_s + n_k < 0) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

Solving for $P(e)$, we obtain :

$$P(e) = \frac{q}{1-p+q} = \frac{Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}{1 - \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}$$

A sketch of the detector structure is shown in the next figure.



Problem 10.15 :

A discrete time transversal filter equivalent to the cascade of the trasmitting filter $g_T(t)$, the channel $c(t)$, the matched filter at the receicer $g_R(t)$ and the sampler, has tap gain coefficients $\{x_m\}$, where :

$$x_m = \begin{cases} 0.9 & m = 0 \\ 0.3 & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The noise ν_k, at the output of the sampler, is a zero-mean Gaussian sequence with autocorrelation function :

$$E[\nu_k \nu_l] = \sigma^2 x_{k-l}, \qquad |k-l| \leq 1$$

If the $\mathcal{Z}$-transform of the sequence $\{x_m\}$, $X(z)$, assumes the factorization :

$$X(z) = F(z)F^*(z^{-1})$$

then the filter $1/F^*(z^{-1})$ can follow the sampler to white the noise sequence ν_k. In this case the output of the whitening filter, and input to the MSE equalizer, is the sequence :

$$u_n = \sum_k I_k f_{n-k} + n_k$$

where n_k is zero mean Gaussian with variance σ^2. The optimum coefficients of the MSE equalizer, c_k, satisfy :

$$\sum_{n=-1}^{1} c_n \Gamma_{kn} = \xi_k, \qquad k = 0, \pm 1$$

where :

$$\Gamma(n-k) = \begin{cases} x_{n-k} + \sigma^2 \delta_{n,k}, & |n-k| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_(k) = \begin{cases} f_{-k}, & -1 \leq k \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

With

$$X(z) = 0.3z + 0.9 + 0.3z^{-1} = (f_0 + f_1 z^{-1})(f_0 + f_1 z)$$

we obtain the parameters f_0 and f_1 as :

$$f_0 = \begin{cases} \pm\sqrt{0.7854} \\ \pm\sqrt{0.1146} \end{cases}, \qquad f_1 = \begin{cases} \pm\sqrt{0.1146} \\ \pm\sqrt{0.7854} \end{cases}$$

The parameters f_0 and f_1 should have the same sign since $f_0 f_1 = 0.3$. However, the sign itself does not play any role if the data are differentially encoded. To have a stable inverse system $1/F^*(z^{-1})$, we select f_0 and f_1 in such a way that the zero of the system $F^*(z^{-1}) = f_0 + f_1 z$ is inside the unit circle. Thus, we choose $f_0 = \sqrt{0.1146}$ and $f_1 = \sqrt{0.7854}$ and therefore, the desired system for the equalizer's coefficients is

$$\begin{pmatrix} 0.9+0.1 & 0.3 & 0.0 \\ 0.3 & 0.9+0.1 & 0.3 \\ 0.0 & 0.3 & 0.9+0.1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.7854} \\ \sqrt{0.1146} \\ 0 \end{pmatrix}$$

Solving this system, we obtain

$$c_{-1} = 0.8596, \quad c_0 = 0.0886, \quad c_1 = -0.0266$$

Problem 10.16 :

(a) The spectrum of the band limited equalized pulse is

$$\begin{aligned} X(f) &= \begin{cases} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x(\frac{n}{2W}) e^{-j\frac{\pi n f}{W}} & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2W} \left[2 + 2\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{W} \left[1 + 1\cos\frac{\pi f}{W}\right] & |f| \leq W \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $W = \frac{1}{2T_b}$

(b) The following table lists the possible transmitted sequences of length 3 and the corresponding output of the detector.

-1	-1	-1	-4
-1	-1	1	-2
-1	1	-1	0
-1	1	1	2
1	-1	-1	-2
1	-1	1	0
1	1	-1	2
1	1	1	4

As it is observed there are 5 possible output levels b_m, with probability $P(b_m = 0) = \frac{1}{4}$, $P(b_m = \pm 2) = \frac{1}{4}$ and $P(b_m = \pm 4) = \frac{1}{8}$.

(c) The transmitting filter $G_T(f)$, the receiving filter $G_R(f)$ and the equalizer $G_E(f)$ satisfy the condition

$$G_T(f)G_R(f)G_E(f) = X(f)$$

The power spectral density of the noise at the output of the equalizer is :

$$\mathcal{S}_\nu(f) = S_n(f)|G_R(f)G_E(f)|^2 = \sigma^2|G_R(f)G_E(f)|^2$$

With

$$G_T(f) = G_R(f) = P(f) = \frac{\pi T_{50}}{2} e^{-\pi T_{50}|f|}$$

the variance of the output noise is :

$$\begin{aligned}
\sigma_\nu^2 &= \sigma^2 \int_{-\infty}^{\infty} |G_R(f)G_E(f)|^2 df = \sigma^2 \int_{-\infty}^{\infty} \left|\frac{X(f)}{G_T(f)}\right|^2 df \\
&= \sigma^2 \int_{-W}^{W} \frac{4}{\pi^2 T_{50}^2 W^2} \frac{|1+\cos\frac{\pi f}{W}|^2}{e^{-2\pi T_{50}|f|}} df \\
&= \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \int_0^W \left(1+\cos\frac{\pi f}{W}\right)^2 e^{2\pi T_{50} f} df
\end{aligned}$$

The value of the previous integral can be found using the formula :

$$\begin{aligned}
&\int e^{ax}\cos^n bx dx \\
&= \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)e^a x \cos^{n-1} bx + n(n-1)b^2 \int e^{ax}\cos^{n-2} bx dx\right]
\end{aligned}$$

Thus, we obtain :

$$\sigma_\nu^2 = \frac{8\sigma^2}{\pi^2 T_{50}^2 W^2} \times \left[\left(e^{2\pi T_{50} W} - 1\right) \left(\frac{1}{2\pi T_{50}} + \frac{2\pi T_{50} + \pi \frac{1}{W^2 T_{50}}}{4\pi^2 T_{50}^2 + 4\frac{\pi^2}{W^2}} \right) - \frac{4\pi T_{50}}{4\pi^2 T_{50}^2 + \frac{\pi^2}{W^2}} \left(e^{2\pi T_{50} W} + 1\right) \right]$$

To find the probability of error using a symbol by symbol detector, we follow the same procedure as in Section 9.3.2. The results are the same with that obtained from a 3-point PAM constellation $(0, \pm 2)$ used with a duobinary signal with output levels having the probability mass function given in part (b). An upper bound of the symbol probability of error is :

$$\begin{aligned} P(e) &< P(|y_m| > 1 | b_m = 0) P(b_m = 0) + 2P(|y_m - 2| > 1 | b_m = 2) P(b_m = 2) \\ &\quad + 2P(y_m + 4 > 1 | b_m = -4) P(b_m = -4) \\ &= P(|y_m| > 1 | b_m = 0) \left[P(b_m = 0) + 2P(b_m = 2) + P(b_m = -4) \right] \\ &= \frac{7}{8} P(|y_m| > 1 | b_m = 0) \end{aligned}$$

But

$$P(|y_m| > 1 | b_m = 0) = \frac{2}{\sqrt{2\pi}\sigma_\nu} \int_1^\infty e^{-x^2/2\sigma_\nu^2} dx$$

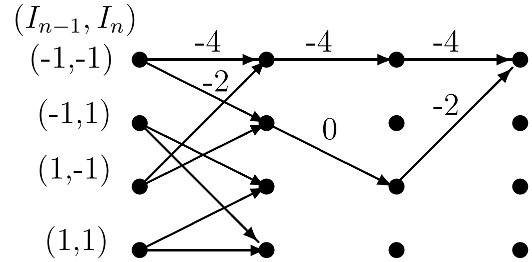
Therefore,

$$P(e) < \frac{14}{8} Q\left[\frac{1}{\sigma_\nu}\right]$$

Problem 10.17 :

Since the partial response signal has memory length equal to 2, the corresponding trellis has 4 states which we label as (I_{n-1}, I_n). The following figure shows three frames of the trellis. The labels of the branches indicate the output of the partial response system. As it is observed the free distance between merging paths is 3, whereas the Euclidean distance is equal to

$$d_E = 2^2 + 4^2 + 2^2 = 24$$



Problem 10.18 :

(a) $X(z) = F(z)F^*(z^{-1}) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1}$. Then, the covariance matrix $\mathbf{\Gamma}$ is :

$$\mathbf{\Gamma} = \begin{bmatrix} 1+N_0 & 1/2 & 0 \\ 1/2 & 1+N_0 & 1/2 \\ 0 & 1/2 & 1+N_0 \end{bmatrix} \text{ and } \xi = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

The optimum equalizer coefficients are given by :

$$\begin{aligned} \mathbf{C}_{opt} &= \mathbf{\Gamma}^{-1}\xi \\ &= \frac{1}{\det(\mathbf{\Gamma})} \begin{bmatrix} (1+N_0)^2 - 1/4 & -\frac{1}{2}(1+N_0) & 1/4 \\ -\frac{1}{2}(1+N_0) & (1+N_0)^2 & -\frac{1}{2}(1+N_0) \\ 1/4 & -\frac{1}{2}(1+N_0) & (1+N_0)^2 - 1/4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}\det(\mathbf{\Gamma})} \begin{bmatrix} N_0^2 + \frac{3}{2}N_0 + \frac{1}{4} \\ N_0^2 + \frac{3}{2}N_0 + \frac{1}{2} \\ -\frac{N_0}{2} - \frac{1}{4} \end{bmatrix} \end{aligned}$$

where $\det(\mathbf{\Gamma}) = (1+N_0)\left[(1+N_0)^2 - \frac{1}{2}\right]$

(b)

$$\begin{aligned} &\det(\mathbf{\Gamma} - \lambda \mathbf{I}) = (1+N_0-\lambda)\left[(1+N_0-\lambda)^2 - \tfrac{1}{2}\right] \Rightarrow \\ &\lambda_1 = 1+N_0,\ \lambda_2 = \tfrac{1}{\sqrt{2}} + 1 + N_0,\ \lambda_3 = 1 - \tfrac{1}{\sqrt{2}} + N_0 \end{aligned}$$

and the corresponding eigenvectors are :

$$\mathbf{v}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

(c)

$$J_{\min}(K)|_{K=1} = J_{\min}(1) = 1 - \xi'\Gamma^{-1}\xi = \frac{2N_0^3 + 4N_0^2 + 2N_0 + 3/4}{2N_0^3 + 4N_0^2 + 5N_0 + 1}$$

(d)

$$\gamma = \frac{1 - J_{\min}(1)}{J_{\min}(1)} = \frac{2N_0^2 + 3N_0 + 3/4}{2N_0^3 + 4N_0^2 + 1/4}$$

Note that as $N_0 \to 0$, $\gamma \to 3$. For $N_0 = 0.1$, $\gamma = 2.18$ for the 3-tap equalizer and $\gamma = \sqrt{1 + \frac{2}{N_0}} - 1 = 3.58$, for the infinite-tap equalizer (as in example 10-2-1). Also, note that $\gamma = \frac{1}{N_0} = 10$ for the case of no intersymbol interference.

Problem 10.19 :

For the DFE we have that :

$$\hat{I}_k = \sum_{j=-K_1}^{0} c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}, \text{ and } \epsilon_k = I_k - \hat{I}_k$$

The orthogonality principle is simply :

$$\left\{ \begin{array}{l} E\left[\epsilon_k u_{k-l}^*\right] = 0, \text{ for } -K_1 \leq l \leq 0 \\ E\left[\epsilon_k I_{k-l}^*\right] = 0, \text{ for } 1 \leq l \leq K_2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} E\left[I_k u_{k-l}^*\right] = E\left[\hat{I}_k u_{k-l}^*\right], \ -K_1 \leq l \leq 0 \\ E\left[I_k I_{k-l}^*\right] = E\left[\hat{I}_k I_{k-l}^*\right], \ 1 \leq l \leq K_2 \end{array} \right\}$$

Since the information symbols are uncorrelated : $E\left[I_k I_l^*\right] = a\delta_{kl}$, where $a = E\left[|I_k|^2\right]$ is a constant whose value is not needed since it will be present in all terms and thus, cancelled out. In order to solve the above system, we also need $E\left[u_k u_l^*\right], \ E\left[I_k u_l^*\right]$. We have :

$$\begin{array}{rcl} E\left[u_k u_l^*\right] & = & E\left[\left(\sum_{n=0}^{L} f_n I_{k-n} + n_k\right)\left(\sum_{m=0}^{L} f_m^* I_{l-m}^* + n_l^*\right)\right] \\ & = & a\sum_{m=0}^{L} f_m^* f_{m+k-l} + N_0\delta_{kl} \end{array}$$

and

$$\begin{array}{rcl} E\left[I_k u_l^*\right] & = & E\left[I_k\left(\sum_{m=0}^{L} f_m^* I_{l-m}^* + n_l^*\right)\right] \\ & = & a f_{l-k}^* \end{array}$$

Hence, the second equation of the orthogonality principle gives :

$$\begin{array}{l} E\left[I_k I_{k-l}^*\right] = E\left[\hat{I}_k I_{k-l}^*\right], \ 1 \leq l \leq K_2 \Rightarrow \\ 0 = E\left[\left(\sum_{j=-K_1}^{0} c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}\right) I_{k-l}^*\right] \Rightarrow \\ 0 = a\left(\sum_{j=-K_1}^{0} c_j f_{l-j}\right) + a c_l \Rightarrow \\ c_l = -\sum_{j=-K_1}^{0} c_j f_{l-j}, \ 1 \leq l \leq K_2 \end{array}$$

which is the desired equation for the feedback taps. The first equation of the orthogonality principle will give :

$$E\left[I_k u_{k-l}^*\right] = E\left[\hat{I}_k u_{k-l}^*\right], \ -K_1 \leq l \leq 0 \Rightarrow$$

$$a f_{-l}^* = E\left[\left(\sum_{j=-K_1}^{0} c_j u_{k-j} + \sum_{j=1}^{K_2} c_j I_{k-j}\right) u_{k-l}^*\right] \Rightarrow$$

$$a f_{-l}^* = \sum_{j=-K_1}^{0} c_j\left(a\sum_{m=0}^{L} f_m^* f_{m+l-j} + N_0\delta_{kl}\right) + a\sum_{j=1}^{K_2} c_j f_{j-l}^*, \ -K_1 \leq l \leq 0$$

Substituting the expression for $c_j, \ 1 \leq j \leq K_2$, that we found above :

$$f_{-l}^* = \sum_{j=-K_1}^{0} c_j\left(\sum_{m=0}^{L} f_m^* f_{m+l-j} + N_0\delta_{kl}\right) - \sum_{j=1}^{K_2}\sum_{m=-K_1}^{0} c_m f_{j-m} f_{j-l}^*, \ -K_1 \leq l \leq 0 \Rightarrow$$

$$f_{-l}^* = \sum_{j=-K_1}^{0} c_j\left(\sum_{m=0}^{L} f_m^* f_{m+l-j} + N_0\delta_{kl}\right) - \sum_{j=-K_1}^{0} c_j \sum_{m=1}^{K_2} f_{m-j} f_{m-l}^*, \ -K_1 \leq l \leq 0 \Rightarrow$$

$$\sum_{j=-K_1}^{0} c_j \psi_{lj} = f_{-l}^*, \ -K_1 \leq l \leq 0$$

where $\psi_{lj} = \sum_{m=0}^{-l} f_m^* f_{m+l-j} + N_0\delta_{lj}$, which is the desired expression for the feedforward taps of the equalizer.

Problem 10.20 :

The tap coefficients for the feedback section of the DFE are given by the equation :

$$\begin{array}{rcl} c_k & = & -\sum_{j=-K_1}^{0} c_j f_{k-j}, \;\; k = 1, 2, ..., K_2 \\ & = & -\left(c_0 f_k + c_{-1} f_{k+1} + ... + c_{-K_1} f_{k+K_1}\right) \end{array}$$

But $f_k = 0$ for $k < 0$ and $k > L$. Therefore :

$$c_L = -c_0 f_L, \;\; c_{L+1} = 0, \;\; c_{L+2} = 0, \;\;\; \text{etc}$$

Problem 10.21 :

(a) The tap coefficients for the feedback section of the DFE are given by the equation : $c_k = -\sum_{j=-K_1}^{0} c_j f_{k-j}, \; 1 \leq k \leq K_2$, and for the feedforward section as the solution to the equations : $\sum_{j=-K_1}^{0} c_j \psi_{lj} = -f_{-l}^*, \; K_1 \leq l \leq 0$. In this case, $K_1 = 1$, and hence : $\sum_{j=-K_1}^{0} c_j \psi_{lj} = -f_{-l}^*, \; l = -1, 0$ or :

$$\begin{array}{c} \psi_{0,0} c_0 + \psi_{0,-1} c_{-1} = f_0^* \\ \psi_{-1,0} c_0 + \psi_{-1,-1} c_{-1} = f_1^* \end{array}$$

But $\psi_{lj} = \sum_{m=0}^{-l} f_m^* f_{m+l-j} + N_0\delta_{lj}$, so the above system can be written :

$$\left[\begin{array}{cc} \frac{1}{2} + N_0 & \frac{1}{2} \\ \frac{1}{2} & 1 + N_0 \end{array}\right] \left[\begin{array}{c} c_0 \\ c_{-1} \end{array}\right] = \left[\begin{array}{c} 1/\sqrt{2} \\ 1/\sqrt{2} \end{array}\right]$$

so :

$$\left[\begin{array}{c} c_0 \\ c_{-1} \end{array}\right] = \frac{1}{\sqrt{2}\left(N_0^2 + \frac{3}{2}N_0 + \frac{1}{4}\right)} \left[\begin{array}{c} \frac{1}{2} + N_0 \\ N_0 \end{array}\right] \approx \left[\begin{array}{c} \sqrt{2} \\ 2\sqrt{2}N_0 \end{array}\right], \text{ for } \mathrm{N}_0 << 1$$

The coefficient for the feedback section is :

$$c_1 = -c_0 f_1 = -\frac{1}{\sqrt{2}} c_0 \approx -1, \text{ for } \mathrm{N}_0 << 1$$

(b)

$$J_{\min}(1) = 1 - \sum_{j=-K_1}^{0} c_j f_{-j} = \frac{2N_0^2 + N_0}{2\left(N_0^2 + \frac{3}{2}N_0 + \frac{1}{4}\right)} \approx 2N_0, \text{ for } \mathrm{N}_0 << 1$$

(c)

$$\gamma = \frac{1 - J_{\min}(1)}{J_{\min}(1)} = \frac{1 + 4N_0}{2N_0(1 + 2N_0)} \approx \frac{1}{2N_0}, \; N_0 << 1$$

(d) For the infinite tap DFE, we have from example 10-3-1 :

$$J_{\min} = \frac{2N_0}{1+N_0+\sqrt{(1+N_0)^2-1}} \approx 2N_0, \; N_0 << 1$$

$$\gamma_\infty = \frac{1-J_{\min}}{J_{\min}} = \frac{1-N_0\sqrt{(1+N_0)^2-1}}{2N_0}$$

(e) For $N_0 = 0.1$, we have :

$$J_{\min}(1) = 0.146, \quad \gamma = 5.83 \text{ (7.66 dB)}$$
$$J_{\min} = 0.128, \quad \gamma_\infty = 6.8 \text{ (8.32 dB)}$$

For $N_0 = 0.01$, we have :

$$J_{\min}(1) = 0.0193, \quad \gamma = 51 \text{ (17.1 dB)}$$
$$J_{\min} = 0.0174, \quad \gamma_\infty = 56.6 \text{ (17.5 dB)}$$

The three-tap equalizer performs very wee compared to the infinite-tap equalizer. The difference in performance is 0.6 dB for $N_0 = 0.1$ and 0.4 dB for $N_0 = 0.01$.

Problem 10.22 :

(a) We have that :

$$\frac{1}{2T} = 900, \; \frac{1+\beta}{2T} = 1200 \Rightarrow$$
$$1 + \beta = 1200/900 = 4/3 \Rightarrow \beta = 1/3$$

(b) Since $1/2T = 900$, the pulse rate $1/T$ is 1800 pulses/sec.

(c) The largest interference is caused by the sequence : $\{1, -1, s, 1, -1, 1\}$ or its opposite in sign. This interference is constructive or destructive depending on the sign of the information symbol s. The peak distortion is $\sum_{k=-2,k\neq 0}^{3} f_k = 1.6$

(d) The probability of the worst-case interference given above is $\left(\frac{1}{2}\right)^5 = 1/32$, and the same is the probability of the sequence that causes the opposite-sign interference.

Problem 10.23 :

(a)

$$F(z) = 0.8 - 0.6z^{-1} \Rightarrow$$
$$X(z) \equiv F(z)F^*(z^{-1}) = (0.8 - 0.6z^{-1})(0.8 - 0.6z) = 1 - 0.48z^{-1} - 0.48z$$

Thus, $x_0 = 1,\ x_{-1} = x_1 = -0.48$.

(b)

$$\frac{1}{T}\sum_{n=-\infty}^{\infty}\left|H\left(\omega + \frac{2\pi n}{T}\right)\right|^2 = X\left(e^{j\omega T}\right) = 1 - 0.48e^{-j\omega T} - 0.48e^{j\omega T} = 1 - 0.96\cos\omega T$$

(c) For the linear equalizer base on the mean-square-error criterion we have :

$$\begin{aligned} J_{\min} &= \tfrac{T}{2\pi}\int_{-\pi/T}^{\pi/T}\tfrac{N_0}{1+N_0-0.96\cos\omega T}d\omega \\ &= \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\tfrac{N_0}{1+N_0-0.96\cos\theta}d\theta \\ &= \tfrac{1}{2\pi}\left(\tfrac{N_0}{1+N_0}\right)\int_{-\pi}^{\pi}\tfrac{1}{1-a\cos\theta}d\theta,\ a = \tfrac{0.96}{1+N_0} \end{aligned}$$

But :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1-a\cos\theta}d\theta = \frac{1}{\sqrt{1-a^2}},\ a^2 < 1$$

Therefore :

$$J_{\min} = \frac{N_0}{1+N_0}\frac{1}{\sqrt{1-\left(\frac{0.96}{1+N_0}\right)^2}} = \frac{N_0}{\sqrt{(1+N_0)^2 - (0.96)^2}}$$

(d) For the decision-feedback equalizer :

$$J_{\min} = \frac{2N_0}{1+N_0+\sqrt{(1+N_0)^2-(0.96)^2}}$$

which follows from the result in example 10.3.1. Note that for $N_0 << 1$,

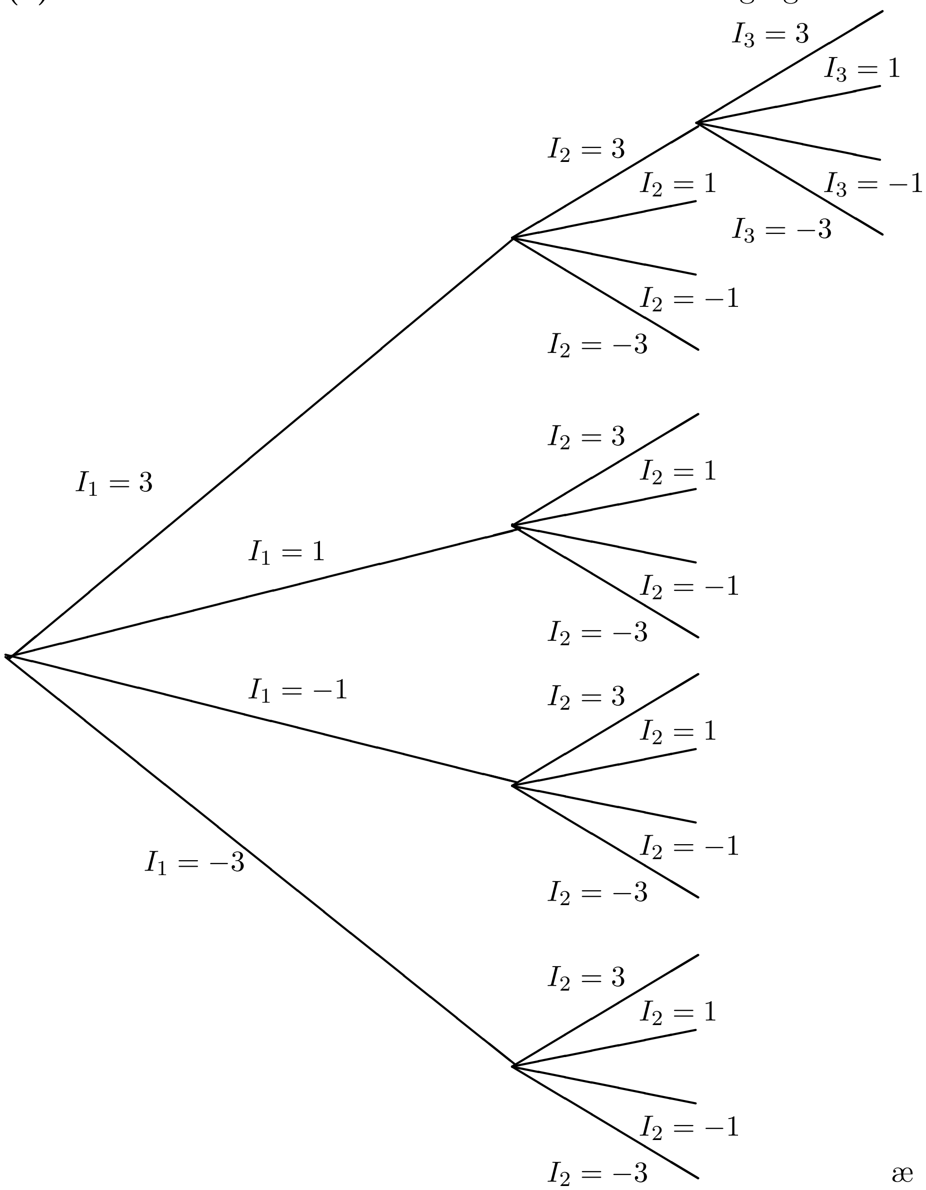
$$J_{\min} \approx \frac{2N_0}{1+\sqrt{1-(0.96)^2}} \approx 1.56N_0$$

In contrast, for the linear equalizer we have :

$$J_{\min} \approx \frac{N_0}{\sqrt{1-(0.96)^2}} \approx 3.57N_0$$

Problem 10.24 :

(a) Part of the tree structure is shown in the following figure :



(b) There are four states in the trellis (corresponding to the four possible values of the symbol I_{k-1}), and for each one there are four paths starting from it (corresponding to the four possible values of the symbol I_k). Hence, 16 probabilities must be computed at each stage of the Viterbi algorithm.

(c) Since, there are four states, the number of surviving sequences is also four.

(d) The metrics are

$$(y_1 - 0.8I_1)^2\,,\; i=1 \;\text{ and } \sum_i (y_i - 0.8I_i + 0.6I_{i-1})^2\,,\; i \geq 2$$

$$\begin{array}{l}
\mu_1\,(I_1=3) = [0.5 - 3*0.8]^2 = 3.61 \\
\mu_1\,(I_1=1) = [0.5 - 1*0.8]^2 = 0.09 \\
\mu_1\,(I_1=-1) = [0.5 + 1*0.8]^2 = 1.69 \\
\mu_1\,(I_1=-3) = [0.5 + 3*0.8]^2 = 8.41
\end{array}$$

$$\begin{array}{l}
\mu_2\,(I_2=3, I_1=3) = \mu_1(3) + [2 - 2.4 + 3*0.6]^2 = 5.57 \\
\mu_2\,(3,1) = \mu_1(1) + [2 - 2.4 + 1*0.6]^2 = 0.13 \\
\mu_2\,(3,-1) = \mu_1(-1) + [2 - 2.4 - 1*0.6]^2 = 6.53 \\
\mu_2\,(3,-3) = \mu_1(-3) + [2 - 2.4 - 3*0.6]^2 = 13.25
\end{array}$$

$$\begin{array}{l}
\mu_2\,(1,3) = \mu_1(3) + [2 - 0.8 + 3*0.6]^2 = 12.61 \\
\mu_2\,(1,1) = \mu_1(1) + [2 - 0.8 + 1*0.6]^2 = 3.33 \\
\mu_2\,(1,-1) = \mu_1(-1) + [2 - 0.8 - 1*0.6]^2 = 2.05 \\
\mu_2\,(1,-3) = \mu_1(-3) + [2 - 0.8 - 3*0.6]^2 = 8.77
\end{array}$$

$$\begin{array}{l}
\mu_2\,(-1,3) = \mu_1(3) + [2 + 0.8 + 3*0.6]^2 = 24.77 \\
\mu_2\,(-1,1) = \mu_1(1) + [2 + 0.8 + 1*0.6]^2 = 11.65 \\
\mu_2\,(-1,-1) = \mu_1(-1) + [2 + 0.8 - 1*0.6]^2 = 6.53 \\
\mu_2\,(-1,-3) = \mu_1(-3) + [2 + 0.8 - 3*0.6]^2 = 9.41
\end{array}$$

$$\begin{array}{l}
\mu_2\,(-3,3) = \mu_1(3) + [2 + 2.4 + 3*0.6]^2 = 42.05 \\
\mu_2\,(-3,1) = \mu_1(1) + [2 + 2.4 + 1*0.6]^2 = 25.09 \\
\mu_2\,(-3,-1) = \mu_1(-1) + [2 + 2.4 - 1*0.6]^2 = 16.13 \\
\mu_2\,(-3,-3) = \mu_1(-3) + [2 + 2.4 - 3*0.6]^2 = 15.17
\end{array}$$

The four surviving paths at this stage are $\min_{I_1}\left[\mu_2(x, I_1)\right],\; x = 3, 1, -1, -3$ or :

$$\begin{array}{l}
I_2 = 3, I_1 = 1 \text{ with metric } \mu_2(3,1) = 0.13 \\
I_2 = 1, I_1 = -1 \text{ with metric } \mu_2(1,-1) = 2.05 \\
I_2 = -1, I_1 = -1 \text{ with metric } \mu_2(-1,-1) = 6.53 \\
I_2 = -3, I_1 = -3 \text{ with metric } \mu_2(-3,-3) = 15.17
\end{array}$$

Now we compute the metrics for the next stage :

$$\begin{array}{l}
\mu_3\,(I_3=3, I_2=3, I_1=1) = \mu_2(3,1) + [-1 - 2.4 + 1.8]^2 = 2.69 \\
\mu_3\,(3,1,-1) = \mu_2(1,-1) + [-1 - 2.4 + 0.6]^2 = 9.89 \\
\mu_3\,(3,-1,-1) = \mu_2(-1,-1) + [-1 - 2.4 - 0.6]^2 = 22.53 \\
\mu_3\,(3,-3,-3) = \mu_2(-3,-3) + [-1 - 2.4 - 1.8]^2 = 42.21
\end{array}$$

$$\begin{array}{l}
\mu_3\left(1,3,1\right)=\mu_2(3,1)+\left[-1-0.8+1.8\right]^2=0.13\\
\mu_3\left(1,1,-1\right)=\mu_2(1,-1)+\left[-1-0.8+0.6\right]^2=7.81\\
\mu_3\left(1,-1,-1\right)=\mu_2(-1,-1)+\left[-1-0.8-0.6\right]^2=12.29\\
\mu_3\left(1,-3,-3\right)=\mu_2(-3,-3)+\left[-1-0.8-1.8\right]^2=28.13
\end{array}$$

$$\begin{array}{l}
\mu_3\left(-1,3,1\right)=\mu_2(3,1)+\left[-1+0.8+1.8\right]^2=2.69\\
\mu_3\left(-1,1,-1\right)=\mu_2(1,-1)+\left[-1+0.8+0.6\right]^2=2.69\\
\mu_3\left(-1,-1,-1\right)=\mu_2(-1,-1)+\left[-1+0.8-0.6\right]^2=7.17\\
\mu_3\left(-1,-3,-3\right)=\mu_2(-3,-3)+\left[-1+0.8-1.8\right]^2=19.17
\end{array}$$

$$\begin{array}{l}
\mu_3\left(-3,3,1\right)=\mu_2(3,1)+\left[-1+2.4+1.8\right]^2=10.37\\
\mu_3\left(-3,1,-1\right)=\mu_2(1,-1)+\left[-1+2.4+0.6\right]^2=2.69\\
\mu_3\left(-3,-1,-1\right)=\mu_2(-1,-1)+\left[-1+2.4-0.6\right]^2=7.17\\
\mu_3\left(-3,-3,-3\right)=\mu_2(-3,-3)+\left[-1+2.4-1.8\right]^2=15.33
\end{array}$$

The four surviving sequences at this stage are $\min_{I_2,I_1}\left[\mu_3(x,I_2,I_1)\right],\ x=3,1,-1,-3$ or :

$$\begin{array}{l}
I_3=3,I_2=3,I_1=1 \text{ with metric } \mu_3(3,3,1)=2.69\\
I_3=1,I_2=3,I_1=1 \text{ with metric } \mu_3(1,3,1)=0.13\\
I_3=-1,I_2=3,I_1=1 \text{ with metric } \mu_3(-1,3,1)=2.69\\
I_3=-3,I_2=1,I_1=-1 \text{ with metric } \mu_3(-3,1,-1)=2.69
\end{array}$$

(e) For the channel, $\delta^2_{\min}=1$ and hence :

$$P_4=8Q\left(\sqrt{\frac{6}{15}\gamma_{av}}\right)$$

Problem 10.25 :

(a)

$$\begin{aligned}
b_k &= \tfrac{1}{K}\textstyle\sum_{n=0}^{K-1}E_n e^{j2\pi nk/K}\\
&= \tfrac{1}{K}\textstyle\sum_{n=0}^{K-1}\sum_{l=0}^{K-1}c_l e^{-j2\pi nl/K}e^{j2\pi nk/K}\\
&= \tfrac{1}{K}\textstyle\sum_{l=0}^{K-1}c_l\sum_{n=0}^{K-1}e^{j2\pi n(k-l)/K}
\end{aligned}$$

But

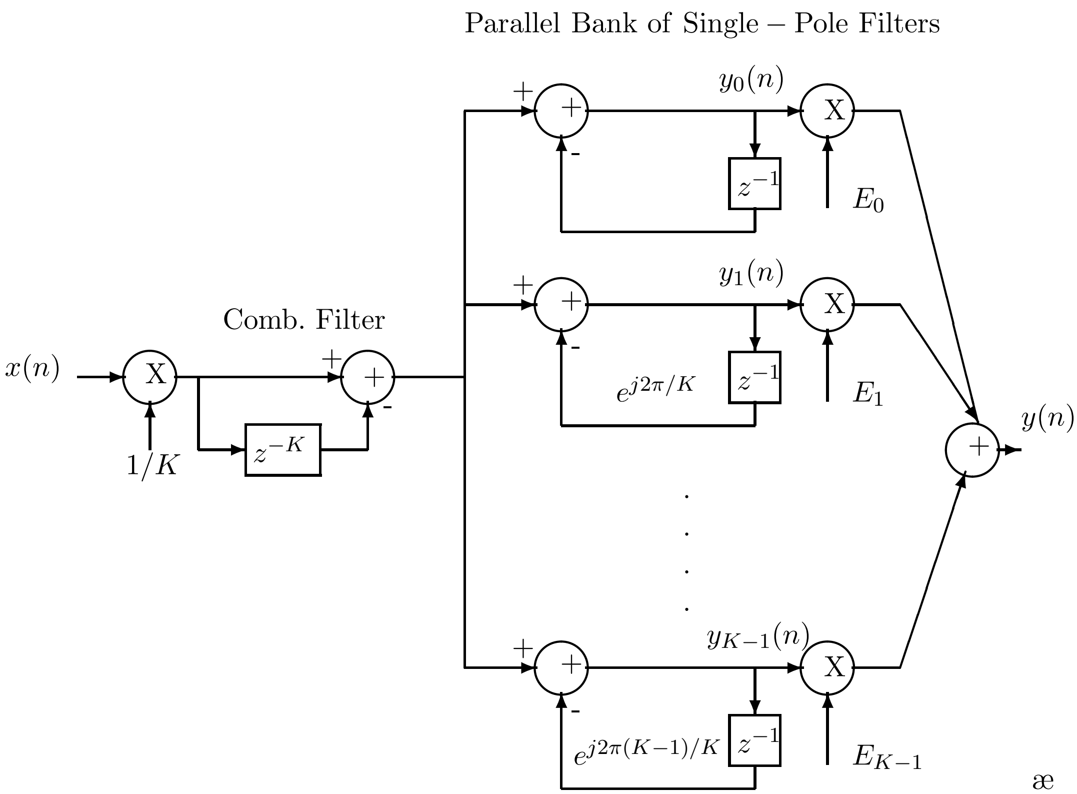
$$\sum_{n=0}^{K-1}e^{j2\pi n(k-l)/K}=\left\{\begin{array}{ll}0, & k\neq l\\ K, & k=l\end{array}\right\}$$

Hence, $b_k=c_k$.

(b)

$$
\begin{aligned}
E(z) &= \sum_{k=0}^{K-1} c_k z^{-k} \\
&= \sum_{k=0}^{K-1} \left[\frac{1}{K} \sum_{n=0}^{K-1} E_n e^{j2\pi nk/K} \right] z^{-k} \\
&= \frac{1}{K} \sum_{n=0}^{K-1} E_n \left[\sum_{k=0}^{K-1} \left(e^{j2\pi n/K} z^{-1} \right)^k \right] \\
&= \frac{1}{K} \sum_{n=0}^{K-1} E_n \frac{1-z^{-K}}{1-\exp(j2\pi n/K)z^{-1}} \\
&= \frac{1-z^{-K}}{K} \sum_{n=0}^{K-1} \frac{E_n}{1-\exp(j2\pi n/K)z^{-1}}
\end{aligned}
$$

(c) The block diagram is as shown in the following figure :



(d) The adjustable parameters in this structure are $\{E_0, E_1, ..., E_{K-1}\}$, i.e. the DFT coefficients of the equalizer taps. For more details on this equalizer structure, see the paper by Proakis (IEEE Trans. on Audio and Electroacc., pp 195-200, June 1970).

CHAPTER 11

Problem 11.1 :

(a)

$$F(z) = \frac{4}{5} + \frac{3}{5}z^{-1} \Rightarrow X(z) = F(z)F^*(z^{-1}) = 1 + \frac{12}{25}\left(z + z^{-1}\right)$$

Hence :

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & \frac{12}{25} & 0 \\ \frac{12}{25} & 1 & \frac{12}{25} \\ 0 & \frac{12}{25} & 1 \end{bmatrix} \quad \xi = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

and :

$$\mathbf{C}_{opt} = \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \mathbf{\Gamma}^{-1}\xi = \frac{1}{\beta}\begin{bmatrix} 1-a^2 & -a & a^2 \\ -a & 1 & -a \\ a^2 & -a & 1-a^2 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix}$$

where $a = 0.48$ and $\beta = 1 - 2a^2 = 0.539$. Hence :

$$\mathbf{C}_{opt} = \begin{bmatrix} 0.145 \\ 0.95 \\ -0.456 \end{bmatrix}$$

(b) The eigenvalues of the matrix $\mathbf{\Gamma}$ are given by :

$$|\mathbf{\Gamma} - \lambda \mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0.48 & 0 \\ 0.48 & 1-\lambda & 0.48 \\ 0 & 0.48 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1,\ 0.3232,\ 1.6768$$

The step size Δ should range between :

$$0 \leq \Delta \leq 2/\lambda_{\max} = 1.19$$

(c) Following equations (10-3-3)-(10-3-4) we have :

$$\psi = \begin{bmatrix} 1 & 0.48 \\ 0.48 & 0.64 \end{bmatrix}, \quad \psi \begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.25 \end{bmatrix}$$

and the feedback tap is :

$$c_1 = -c_0 f_1 = -0.75$$

Problem 11.2 :

(a)

$$\Delta_{\max} = \frac{2}{\lambda_{\max}} = \frac{2}{1 + \frac{1}{\sqrt{2}} + N_0} = \frac{2}{1.707 + N_0}$$

(b) From (11-1-31) :

$$J_\Delta = \Delta^2 J_{\min} \sum_{k=1}^{3} \frac{\lambda_k^2}{1 - (1 - \Delta\lambda_k)^2} \approx \frac{1}{2} \Delta J_{\min} \sum_{k=1}^{3} \lambda_k$$

Since $\frac{J_\Delta}{J_{\min}} = 0.01$:

$$\Delta \approx \frac{0.07}{1 + N_0} \approx 0.06$$

(c) Let $\mathbf{C}' = \mathbf{V}^t \mathbf{C}$, $\xi' = \mathbf{V}^t \xi$, where $\mathbf{V}$ is the matrix whose columns form the eigenvectors of the covariance matrix $\mathbf{\Gamma}$ (note that $\mathbf{V}^t = \mathbf{V}^{-1}$). Then :

$$\begin{aligned}
&\mathbf{C}_{(n+1)} = (\mathbf{I} - \Delta\mathbf{\Gamma})\,\mathbf{C}_{(n)} + \Delta\xi \Rightarrow \\
&\mathbf{C}_{(n+1)} = \left(\mathbf{I} - \Delta\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)\mathbf{C}_{(n)} + \Delta\xi \Rightarrow \\
&\mathbf{V}^{-1}\mathbf{C}_{(n+1)} = \mathbf{V}^{-1}\left(\mathbf{I} - \Delta\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)\mathbf{C}_{(n)} + \Delta\mathbf{V}^{-1}\xi \Rightarrow \\
&\mathbf{C}'_{(n+1)} = (\mathbf{I} - \Delta\mathbf{\Lambda})\,\mathbf{C}'_{(n)} + \Delta\xi'
\end{aligned}$$

which is a set of three de-coupled difference equations (de-coupled because Λ is a diagonal matrix). Hence, we can write :

$$c'_{k,(n+1)} = (1 - \Delta\lambda_k)\, c'_{k,(n)} + \Delta\xi'_k, \quad k = -1, 0, 1$$

The steady-state solution is obtained when $c'_{k,(n+1)} = c'_k$, which gives :

$$c'_k = \frac{\xi'_k}{\lambda_k}, \quad k = -1, 0, 1$$

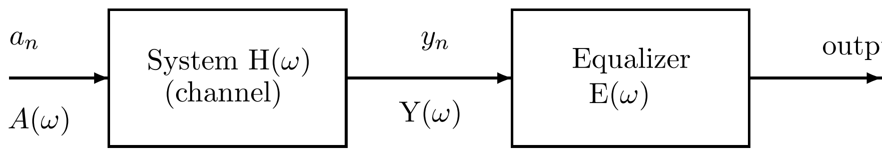
or going back to matrix form :

$$\begin{aligned}
&\mathbf{C}' = \mathbf{\Lambda}^{-1}\xi' \Rightarrow \\
&\mathbf{C} = \mathbf{V}\mathbf{C}' = \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{-1}\xi \Rightarrow \\
&\mathbf{C} = \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)^{-1}\xi = \mathbf{\Gamma}^{-1}\xi
\end{aligned}$$

which agrees with the result in Probl. 10.18(a).

Problem 11.3 :

Suppose that we have a discrete-time system with frequency response $H(\omega)$; this may be equalized by use of the DFT as shown below :



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$$A(\omega) = \sum_{n=0}^{N-1} a_n e^{-j\omega n} \quad Y(\omega) = \sum_{n=0}^{N-1} c_n e^{-j\omega n} = A(\omega)H(\omega)$$

Let :

$$E(\omega) = \frac{A(\omega)Y^*(\omega)}{|Y(\omega)|^2}$$

Then by direct substitution of $Y(\omega)$ we obtain :

$$E(\omega) = \frac{A(\omega)A^*(\omega)H^*(\omega)}{|A(\omega)|^2 |H(\omega)|^2} = \frac{1}{H(\omega)}$$

If the sequence $\{a_n\}$ is sufficiently padded with zeros, the N-point DFT simply represents the values of $E(gw)$ and $H(\omega)$ at $\omega = \frac{2\pi}{N}k = \omega_k$, for $k = 0, 1, ...N-1$ without frequency aliasing. Therefore the use of the DFT as specified in this problem yields $E(\omega_k) = \frac{1}{H(\omega)}$, independent of the properties of the sequence $\{a_n\}$. Since H(ω) is the spectrum of the discrete-time system, we know that this is equivalent to the folded spectrum of the continuous-time system (i.e the system which was sampled). For further details for the use of a pseudo-random periodic sequence to perform equalization we refer to the paper by Qureshi (1985).

Problem 11.4 :

The MSE performance index at the time instant k is

$$J(\mathbf{c}_k) = E\left[\left|\sum_{n=-N}^{N} c_{k,n}v_{k-n} - I_k\right|^2\right]$$

If we define the gradient vector $\mathbf{G}_k$ as

$$\mathbf{G}_k = \frac{\vartheta J(\mathbf{c}_k)}{2\vartheta \mathbf{c}_k}$$

then its $l-th$ element is

$$G_{k,l}=\frac{\vartheta J(\mathbf{c}_k)}{2\vartheta c_{k,l}} = \frac{1}{2}E\left[2\left(\sum_{n=-N}^{N} c_{k,n}v_{k-n}-I_k\right)v_{k-l}^*\right]$$
$$= E\left[-\epsilon_k v_{k-l}^*\right] = -E\left[\epsilon_k v_{k-l}^*\right]$$

Thus, the vector $\mathbf{G}_k$ is

$$\mathbf{G}_k=\begin{pmatrix} -E[\epsilon_k v_{k+N}^*] \\ \vdots \\ -E[\epsilon_k v_{k-N}^*] \end{pmatrix} = -E[\epsilon_k \mathbf{V}_k^*]$$

where $\mathbf{V}_k$ is the vector $\mathbf{V}_k=[v_{k+N}\cdots v_{k-N}]^T$. Since $\hat{\mathbf{G}}_k=-\epsilon_k\mathbf{V}_k^*$, its expected value is

$$E[\hat{\mathbf{G}}_k]=E[-\epsilon_k\mathbf{V}_k^*]=-E[\epsilon_k\mathbf{V}_k^*]=\mathbf{G}_k$$

Problem 11.5 :

The tap-leakage LMS algorithm is :

$$\mathbf{C}(n+1)=w\mathbf{C}(n)+\Delta\epsilon(n)\mathbf{V}^*(n)=w\mathbf{C}(n)+\Delta\left(\mathbf{\Gamma}\mathbf{C}(n)-\xi\right)=\left(w\mathbf{I}-\Delta\mathbf{\Gamma}\right)\mathbf{C}(n)-\Delta\xi$$

Following the same diagonalization procedure as in Problem 11.2 or Section (11-1-3) of the book, we obtain :

$$\mathbf{C}'(n+1)=\left(w\mathbf{I}-\Delta\mathbf{\Lambda}\right)\mathbf{C}'(n)-\Delta\xi'$$

where $\mathbf{\Lambda}$ is the diagonal matrix containing the eigenvalues of the correlation matrix $\mathbf{\Gamma}$. The algorithm converges if the roots of the homogeneous equation lie inside the unit circle :

$$|w-\Delta\lambda_k|<1, \quad k=-N,...,-1,0,1,...,N$$

and since $\Delta>0$, the convergence criterion is :

$$\Delta<\frac{1+w}{\lambda_{\max}}$$

Problem 11.6 :

The estimate of g can be written as : $\hat{g}=h_0x_0+...+h_{M-1}x_{M-1}=\mathbf{x}^T\mathbf{h}$, where $\mathbf{x},\mathbf{h}$ are column vectors containing the respective coefficients. Then using the orthogonality principle we obtain the optimum linear estimator $\mathbf{h}$:

$$E\left[\mathbf{x}\epsilon\right]=0\Rightarrow E\left[\mathbf{x}\left(g-\mathbf{x}^T\mathbf{h}\right)\right]=0\Rightarrow E\left[\mathbf{x}g\right]=E\left[\mathbf{x}\mathbf{x}^T\right]\mathbf{h}$$

or :

$$\mathbf{h}_{opt} = \mathbf{R}_{xx}^{-1}\mathbf{c}$$

where the $M \times M$ correlation matrix $\mathbf{R}_{xx}$ has elements :

$$R(m,n) = E[x(m)x(n)] = E\left[g^2\right]u(m)u(n) + \sigma_w^2\delta_{nm} = Gu(m)u(n) + \sigma_w^2\delta_{nm}$$

where we have used the fact that g and w are independent, and that $E[g] = 0$. Also, the column vector $\mathbf{c} = E[\mathbf{x}g]$ has elements :

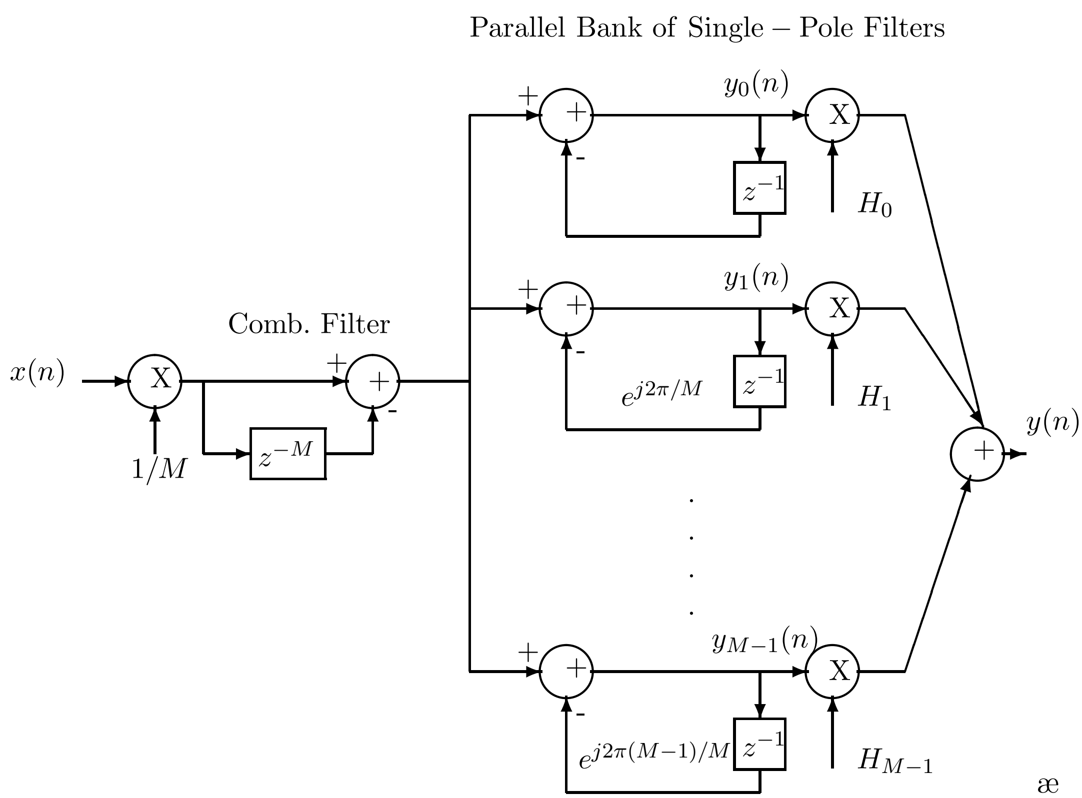
$$c(n) = E[x(n)g] = Gu(n)$$

Problem 11.7 :

(a) The time-update equation for the parameters $\{H_k\}$ is :

$$H_k^{(n+1)} = H_k^{(n)} + \Delta\epsilon^{(n)}y_k^{(n)}$$

where n is the time-index, k is the filter index, and $y_k^{(n)}$ is the output of the k-th filter with transfer function : $\left(1 - z^{-M}\right) / \left(1 - e^{j2\pi k/M}z^{-1}\right)$ as shown in the figure below :



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The error $\epsilon(n)$ is calculated as : $\epsilon(n) = I_n - y(n)$, and then it is fed back in the adaptive part of the equalizer, together with the quantities $y_k^{(n)}$, to update the equalizer parameters H_k.

(b) It is straightforward to prove that the transfer function of the k-th filter in the parallel bank has a resonant frequency at $f_k = 2\pi\frac{k}{M}$, and is zero at the resonant frequencies of the other filters $f_m = 2\pi\frac{m}{M},\ m \neq k$. Hence, if we choose as a test signal sinusoids whose frequencies coincide with the resonant frequencies of the tuned circuits, this allows the coefficient H_k for each filter to be adjusted independently without any interaction from the other filters.

Problem 11.8 :

(a) The gradient of the performance index J with respect to h is : $\frac{dJ}{dh} = 2h + 40$. Hence, the time update equation becomes :

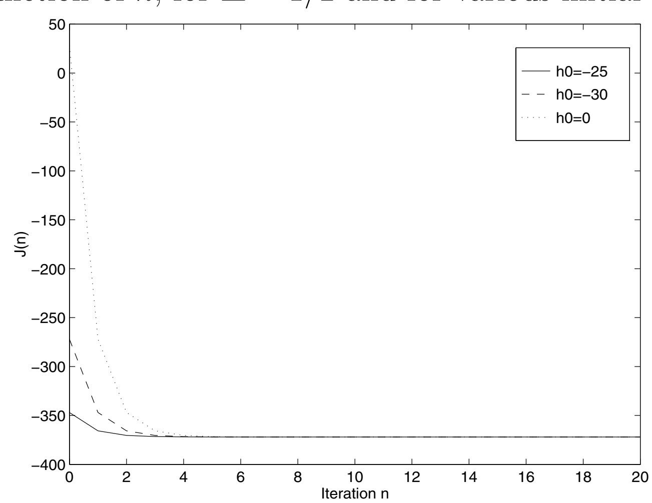
$$h_{n+1} = h_n - \frac{1}{2}\Delta(2h_n + 40) = h_n(1-\Delta) - 20\Delta$$

This system will converge if the homogeneous part will vanish away as $n \to \infty$, or equivalently if : $|1-\Delta| < 1 \Longleftrightarrow 0 < \Delta < 2$.

(b) We note that J has a minimum at $h = -20$, with corresponding value : $J_{\min} = -372$. To illustrate the convergence of the algorithm let's choose : $\Delta = 1/2$. Then : $\mathrm{h}_{n+1} = h_n/2 - 10$, and, using induction, we can prove that :

$$h_{n+1} = \left(\frac{1}{2}\right)^n h_0 - 10\left[\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^k\right]$$

where h_0 is the initial value for h. Then, as $n \to \infty$, the dependence on the initial condition h_0 vanishes and $h_n \to -10\frac{1}{1-1/2} = -20$, which is the desired value. The following plot shows the expression for J as a function of n, for $\Delta = 1/2$ and for various initial values h_0.



Problem 11.9 :

The linear estimator for x can be written as : $\hat{x}(n) = a_1 x(n-1) + a_2 x(n-1) = [x(n-1)\ x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Using the orthogonality principle we obtain :

$$E\left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} \epsilon \right\} = 0 \Rightarrow E\left\{ \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} \left(x(n) - [x(n-1)\ x(n-2)] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) \right\} = 0$$

or :

$$\begin{bmatrix} \gamma_{xx}(-1) \\ \gamma_{xx}(-2) \end{bmatrix} = \begin{bmatrix} \gamma_{xx}(0) & \gamma_{xx}(1) \\ \gamma_{xx}(-1) & \gamma_{xx}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}^{-1} \begin{bmatrix} b \\ b^2 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

This is a well-known fact from Statistical Signal Processing theory : a first-order AR process (which has autocorrelation function $\gamma(m) = a^{|m|}$) has a first-order optimum (MSE) linear estimator : $\hat{x}_n = a x_{n-1}$.

Problem 11.10 :

In Probl. 11.9 we found that the optimum (MSE) linear predictor for $x(n)$, is $\hat{x}(n) = bx(n-1)$. Since it is a first order predictor, the corresponding lattice implementation will comprise of one stage, too, with reflection coefficient a_{11}. This coefficient can be found using (11-4-28) :

$$a_{11} = \frac{\gamma_{xx}(1)}{\gamma_{xx}(0)} = b$$

Then, we verify that the residue $f_1(n)$ is indeed the first-order prediction error : $f_1(n) = x(n) - bx(n-1) = x(n) - \hat{x}(n) = e(n)$

Problem 11.11 :

The system $C(z) = \frac{1}{1-0.9z^{-1}}$ has an impulse response : $c(n) = (0.9)^n,\ n \geq 0$. Then, we write the input $y(n)$ to the adaptive FIR filter :

$$y(n) = \sum_{k=0}^{\infty} c(k) x(n-k) + w(n)$$

Since the sequence $\{x(n)\}$ corresponds to the information sequence that is transmitted through a channel, we will assume that is uncorrelated with zero mean and unit variance. Then the optimum (according to the MSE criterion) estimator of $x(n)$ will be : $\hat{x}(n) = [y(n) \quad y(n-1)] \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$.

Using the orthogonality criterion we obtain the optimum coefficients $\{b_i\}$:

$$E\left\{\left[\begin{array}{c} y(n) \\ y(n-1) \end{array}\right]\epsilon\right\}=0 \Rightarrow E\left\{\left[\begin{array}{c} y(n) \\ y(n-1) \end{array}\right]\left(x(n)-[y(n)\ \ y(n-1)]\left[\begin{array}{c} b_0 \\ b_1 \end{array}\right]\right)\right\}=0$$

$$\Rightarrow \left[\begin{array}{c} b_0 \\ b_1 \end{array}\right]=\left\{E\left[\begin{array}{cc} y(n)y(n) & y(n)y(n-1) \\ y(n-1)y(n) & y(n-1)y(n-1) \end{array}\right]\right\}^{-1}\left\{E\left[\begin{array}{c} y(n)x(n) \\ y(n-1)x(n) \end{array}\right]\right\}$$

The various correlations are as follows :

$$E\left[y(n)x(n)\right]=E\left[\sum_{k=0}^{\infty}c(k)x(n-k)x(n)+w(n)x(n)\right]=c(0)=1$$

where we have used the fact that : $E\left[x(n-k)x(n)\right]=\delta_k$, and that $\{w(n)\}$ $\{x(n)\}$ are independent. Similarly :

$$E\left[y(n-1)x(n)\right]=E\left[\sum_{k=0}^{\infty}c(k)x(n-k-1)x(n)+w(n)x(n)\right]=0$$

$$\begin{aligned} E\left[y(n)y(n)\right] &= E\left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c(k)c(j)x(n-k)x(n-j)\right]+\sigma_w^2 \\ &= \sum_{j=0}^{\infty}c(j)c(j)+\sigma_w^2=\sum_{j=0}^{\infty}(0.9)^{2j}+\sigma_w^2= \\ &= \frac{1}{1-0.81}+\sigma_w^2=\frac{1}{0.19}+\sigma_w^2 \end{aligned}$$

and :

$$\begin{aligned} E\left[y(n)y(n-1)\right] &= E\left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c(k)c(j)x(n-k)x(n-1-j)\right] \\ &= \sum_{j=0}^{\infty}c(j)c(j+1)=\sum_{j=0}^{\infty}(0.9)^{2j+1} \\ &= 0.9\frac{1}{1-0.81}=0.9\frac{1}{0.19} \end{aligned}$$

Hence :

$$\left[\begin{array}{c} b_0 \\ b_1 \end{array}\right]=\left[\begin{array}{cc} \frac{1}{0.19}+0.1 & 0.9\frac{1}{0.19} \\ 0.9\frac{1}{0.19} & \frac{1}{0.19}+0.1 \end{array}\right]^{-1}\left[\begin{array}{c} 1 \\ 0 \end{array}\right]=\left[\begin{array}{c} 0.85 \\ -0.75 \end{array}\right]$$

It is interesting to note that in the absence of noise (i.e when the term $\sigma_w^2=0.1$ is missing from the diagonal of the correlation matrix), the optimum coefficients are : $B(z)=b_0+b_1z^{-1}=1-0.9z^{-1}$, i.e. the equalizer function is the inverse of the channel function (in this case the MSE criterion coincides with the zero-forcing criterion). However, we see that, in the presence of noise, the MSE criterion gives a slightly different result from the inverse channel function, in order to prevent excessive noise enhancement.

Problem 11.12 :

(a) If we denote by $\mathbf{V}$ the matrix whose columns are the eigenvectors $\{\mathbf{v}_i\}$:

$$\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2|...|\mathbf{v}_N]$$

then its conjugate transpose matrix is :

$$\mathbf{V}^{*t} = \begin{bmatrix} \mathbf{v}_1^{*t} \\ \mathbf{v}_2^{*t} \\ \cdots \\ \mathbf{v}_N^{*t} \end{bmatrix}$$

and $\mathbf{\Gamma}$ can be written as :

$$\mathbf{\Gamma} = \sum_{i=1}^{N} \lambda_i \mathbf{v_i} \mathbf{v_i^{*t}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{*t}$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{\Gamma}$. Then, if we name $\mathbf{X} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t}$, we see that :

$$\mathbf{X}\mathbf{X} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t}\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{*t} = \mathbf{\Gamma}$$

where we have used the fact that the matrix $\mathbf{V}$ is unitary : $\mathbf{V}\mathbf{V}^{*t} = \mathbf{I}$. Hence, since $\mathbf{X}\mathbf{X} = \mathbf{\Gamma}$, this shows that the matrix $\mathbf{X} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t} = \sum_{i=1}^{N} \lambda_i^{1/2} \mathbf{v_i} \mathbf{v_i^{*t}}$ is indeed the square root of $\mathbf{\Gamma}$.

(b) To compute $\mathbf{\Gamma}^{1/2}$, we first determine $\mathbf{V}, \mathbf{\Lambda}$ (i.e the eigenvalues and eigenvectors of the correlation matrix). Then :

$$\mathbf{\Gamma}^{1/2} = \sum_{i=1}^{N} \lambda_i^{1/2} \mathbf{v_i} \mathbf{v_i^{*t}} = \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^{*t}$$

CHAPTER 12

Problem 12.1 :

(a)

$$\begin{aligned}
U &= \sum_{n=1}^{N} X_n \\
E[U] &= \sum_{n=1}^{N} E[X_n] = Nm \\
\sigma_u^2 &= E\left[U^2\right] - E^2[U] = E\left[\sum_n \sum_m X_n X_m\right] - N^2 m^2 \\
&= N\left(\sigma^2 + m^2\right) + N(N-1)\sigma^2 - N^2 m^2 = N\sigma^2
\end{aligned}$$

Hence :

$$(SNR)_u = \frac{N^2 m^2}{2N\sigma^2} = \frac{N}{2}\frac{m^2}{\sigma^2}$$

(b)

$$\begin{aligned}
V &= \sum_{n=1}^{N} X_n^2 \\
E[V] &= \sum_{n=1}^{N} E\left[X_n^2\right] = N\left(\sigma^2 + m^2\right)
\end{aligned}$$

For the variance of V we have :

$$\sigma_V^2 = E\left[V^2\right] - E^2[V] = E\left[V^2\right] - N^2\left(\sigma^2 + m^2\right)$$

But :

$$E\left[V^2\right] = \sum_n \sum_m E\left(X_n^2 X_m^2\right) = \sum_{n=1}^{N} X_n^4 + \sum_n \sum_{m,n\neq m} E\left(X_n^2\right) E\left(X_m^2\right) = NE\left(X^4\right) + N(N-1)E^2\left(X^2\right)$$

To compute $E\left(X^4\right)$ we can use the fact that a zero-mean Gaussian RV Y has moments :

$$E\left[Y^k\right] = \left\{ \begin{array}{cc} 0, & \text{k : odd} \\ 1 \cdot 3 \cdot ...(k-1)\sigma^k & \text{k : even} \end{array} \right\}$$

Hence :

$$\left\{ \begin{array}{c} E\left[(X-m)^3\right] = 0 \\ E\left[(X-m)^4\right] = 3\sigma^4 \end{array} \right\} \Rightarrow E\left[X^4\right] = m^4 + 6\sigma^2 m^2 + 3\sigma^4$$

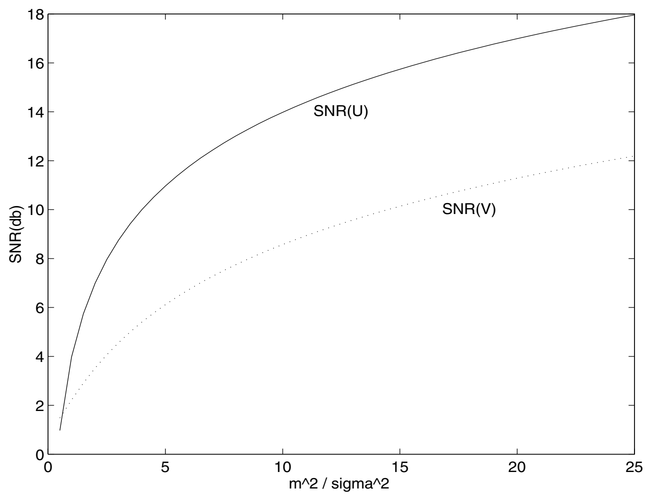
Then :

$$E\left[V^2\right] = N\left(m^4 + 6\sigma^2 m^2 + 3\sigma^4\right) + N(N-1)\left(\sigma^2 + m^2\right) \Rightarrow$$
$$\sigma_V^2 = E\left[V^2\right] - N^2\left(\sigma^2 + m^2\right) = 2N\sigma^2\left(\sigma^2 + 2m^2\right)$$

Note : the above result could be obtained by noting that V is a non-central chi-square RV, with N degrees of freedom and non-centrality parameter equal to Nm^2; then we could apply directly expression (2-1-125). Having obtained σ_V^2, we have :

$$(SNR)_V = \frac{N^2\left(m^2 + \sigma^2\right)^2}{2N\sigma^2\left(\sigma^2 + 2m^2\right)} = \frac{N}{4}\frac{\left((m^2/\sigma^2) + 1\right)^2}{\left(2\left(m^2/\sigma^2\right) + 1\right)}$$

(c) The plot is given in the following figure for N=5 :



(d) In multichannel operation with coherent detection the decision variable is U as given in (a). With square-law detection, the decision variable is of the form $\sum_{n=1}^{N}\left|X_n + jY_n\right|^2$ where X_n and Y_n are Gaussian. We note that V is not the exact model for square-law detection, but, nevertheless, the effect of the non coherent combining loss is evident in the $(SNR)_V$.

Problem 12.2 :

(a) r is a Gaussian random variable. If $\sqrt{\mathcal{E}_b}$ is the transmitted signal point, then :

$$E(r) = E(r_1) + E(r_2) = (1+k)\sqrt{\mathcal{E}_b} \equiv m_r$$

and the variance is :

$$\sigma_r^2 = \sigma_1^2 + k^2\sigma_2^2$$

The probability density function of r is

$$p(r) = \frac{1}{\sqrt{2\pi}\sigma_r}e^{-\frac{(r-m_r)^2}{2\sigma_r^2}}$$

and the probability of error is :

$$P_2 = \int_{-\infty}^{0} p(r)\,dr = Q\left(\sqrt{\frac{m_r^2}{\sigma_r^2}}\right)$$

where

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+k)^2\mathcal{E}_b}{\sigma_1^2 + k^2\sigma_2^2}$$

The value of k that maximizes this ratio is obtained by differentiating this expression and solving for the value of k that forces the derivative to zero. Thus, we obtain

$$k = \frac{\sigma_1^2}{\sigma_2^2}$$

Note that if $\sigma_1 > \sigma_2$, then $k > 1$ and r_2 is given greater weight than r_1. On the other hand, if $\sigma_2 > \sigma_1$, then $k < 1$ and r_1 is given greater weight than r_2. When $\sigma_1 = \sigma_2$, $k = 1$ (equal weight).

(b) When $\sigma_2^2 = 3\sigma_1^2$, $k = \frac{1}{3}$, and

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+\frac{1}{3})^2\mathcal{E}_b}{\sigma_1^2 + \frac{1}{9}(3\sigma_1^2)} = \frac{4}{3}\left(\frac{\mathcal{E}_b}{\sigma_1^2}\right)$$

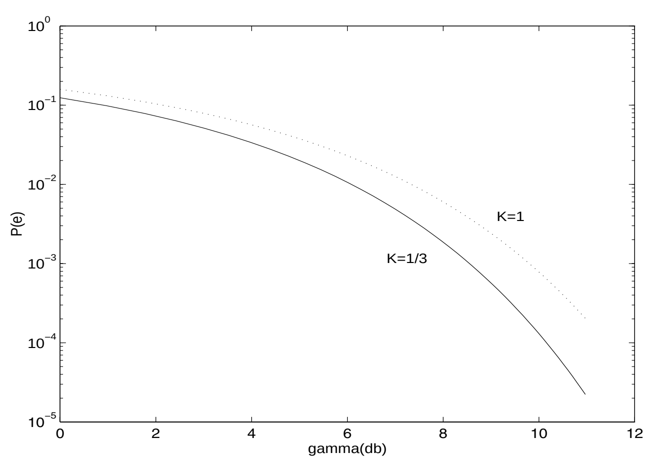
On the other hand, if k is set to unity we have

$$\frac{m_r^2}{\sigma_r^2} = \frac{4\mathcal{E}_b}{\sigma_1^2 + 3\sigma_1^2} = \frac{\mathcal{E}_b}{\sigma_1^2}$$

Therefore, the optimum weighting provides a gain of :

$$10\log\frac{4}{3} = 1.25 \text{ dB}$$

This is illustrated in the following figure, where $\gamma = \frac{\mathcal{E}_b}{\sigma_1^2}$.



Problem 12.3 :

(a) If the sample rate $\frac{1}{T_s} = \tilde{N} \cdot \Delta f = W$, does not alter with the insertion of the cyclic prefix (which indeed is the case in most multicarrier systems), then the bandwidth requirements for the system remain the same. However, keeping the same sample rate means that the block length is increased by a factor of $\frac{v}{N}$, and the effective throughput is reduced to $\frac{1}{1+\frac{v}{N}} = \frac{N}{N+v}$ of the previous one. This is usually compensated by the elimination of ISI, which allows the use of higher order alphabets in each one of the subcarriers.
If the sample rate is increased by a factor of $\left(\frac{N}{N+v}\right)^{-1}$, so that the block length after the insertion of the cyclic prefix will be the same as before, then the bandwidth requirements for the system are increased by the same factor : $W' = W\frac{N+v}{N}$. However, this second case is rarely used in practice.

(b) If the real and imaginary parts of the information sequence $\{X_k\}$ have the same average energy : $E\left[Re(X_k)\right]^2 = E\left[Im(X_k)\right]^2$, then it is straightforward to prove that the time-domain samples $\{x_n\}$, that are the output of the IDFT, have the same average energy:

$$x_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(Re(X_k) + j \cdot Im(X_k)\right) \exp(j2\pi nk/N), \quad n = 0, 1, ..., N-1$$

and :

$$E\left[x_n^2\right] = \epsilon$$

for all $n = 0, 1, ...N-1$. Hence, the energy of the cyclic-prefixed block, will be increased from $N\epsilon$ to $(N+v)\,\epsilon$. However, the power requirements will remain the same, since the duration of the prefixed block is also increased from NT_s to $(N+v)\,T_s$.
For an analysis of the case where the real and imaginary parts of the information sequence do not have the same average energy, we refer the interested reader to the paper by Chow et al. (1991).

Problem 12.4 :

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}, \quad k = 0, ..., N-1$$

and for the padded sequence :

$$X'(k) = \sum_{n=0}^{N+L-1} x'(n) e^{-j2\pi nk/(N+L)} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/(N+L)}, \quad k = 0, ..., N+L-1$$

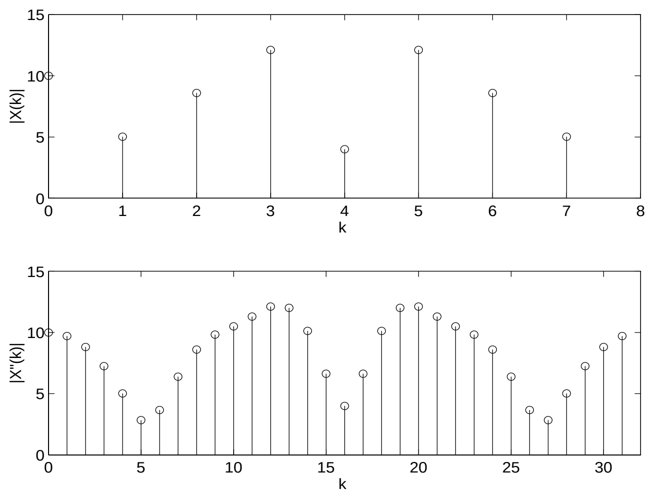
where we have used the fact that : $x'(n) = 0,\ n = N, N+1, ..., N+L-1$. We have also chosen to use the traditional definition of the DFT (without a scaling factor in front of the sum). Then:

$$X(0) = \sum_{n=0}^{N-1} x(n) = X'(0)$$

If we plot $|X(k)|$ and $|X'(k)|$ in the same graph, with the x-axis being the normalized frequency $f = \frac{k}{N}$ or $f = \frac{k}{N+L}$, respectively, then we notice that the second graph is just an interpolated version of the first. This can be seen if $N+L$ is an integer multiple of $N : N+L = mN$. Then :

$$X'(mk) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nmk/mN} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = X(k), \quad k = 0, 1, ...N-1$$

This is illustrated in the following plot, for a random sequence $x(n)$, of length $N = 8$, which is padded with $L = 24$ zeros.



Problem 12.5 :

The analog signal is :

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kt/T}, \quad 0 \le t < T$$

The subcarrier frequencies are : $F_k = k/T,\ k = 0, 1, ...\tilde{N}$, and, hence, the maximum frequency in the analog signal is : $\tilde{N}/T$. If we sample at the Nyquist rate : $2\tilde{N}/T = N/T$, we obtain the discrete-time sequence :

$$x(n) = x(t = nT/N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi k(nT/N)/T} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 1, ..., N-1$$

which is simply the IDFT of the information sequence $\{X_k\}$.

Problem 12.6 :

The reseting of the filter state every N samples, is equivalent to a filter with system function :

$$H_n(z) = \frac{1 - z^{-N}}{1 - \exp(j2\pi n/N)z^{-1}}$$

We will make use of the relationship that gives the sum of finite geometric series : $\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a}$. Using this we can re-write each system function $H_n(z)$ as :

$$H_n(z) = \frac{1 - z^{-N}}{1 - \exp(j2\pi n/N)z^{-1}} = \frac{1 - [\exp(j2\pi n/N)z^{-1}]^N}{1 - \exp(j2\pi n/N)z^{-1}} = \sum_{k=0}^{N-1} \left(\exp(j2\pi n/N)z^{-1}\right)^k$$

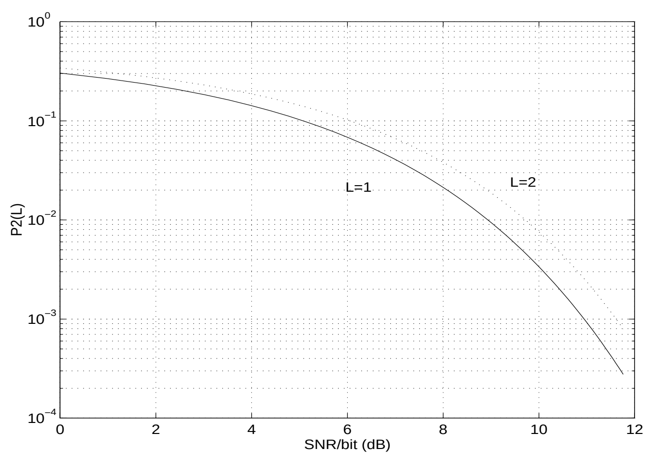
or :

$$H_n(z) = \sum_{k=0}^{N-1} \exp(j2\pi nk/N)z^{-k}, \quad n = 0, 1, ...N - 1$$

which is exactly the transfer function of the transversal filter which calculates the n-th IDFT point of a sequence.

Problem 12.7 :

We assume binary ($M = 2$) orthogonal signalling with square-law detection (DPSK signals wil have the same combining loss). Using (12-1-24) and (12-1-14) we obtain the following graph for $P_2(L)$, where SNR/bit $= 10log_{10}\gamma_b$:



From this graph we note that the combining loss for $\gamma_b = 10$ is approximately 0.8 dB.

CHAPTER 13

Problem 13.1 :

$$g(t)=\sqrt{\frac{16\mathcal{E}_c}{3T_c}}\cos^2\frac{\pi}{T_c}\left(t-\frac{T_c}{2}\right), \quad 0\leq t\leq T_c$$

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \\ &= \sqrt{\frac{16\mathcal{E}_c}{3T_c}}\int_0^{T_c}\cos^2\frac{\pi}{T_c}\left(t-\frac{T_c}{2}\right)e^{-j2\pi ft}dt \end{aligned}$$

But $\cos^2\frac{\pi}{T_c}\left(t-\frac{T_c}{2}\right)=\frac{1}{2}\left[1+\cos\frac{2\pi}{T_c}\left(t-\frac{T_c}{2}\right)\right]$. Then

$$G(0)=\frac{1}{2}\sqrt{\frac{16\mathcal{E}_c}{3T_c}}T_c=\sqrt{\frac{4\mathcal{E}_cT_c}{3}}\Rightarrow|G(0)|^2=\frac{4\mathcal{E}_cT_c}{3}$$

and

$$\sigma_m^2=4w_mJ_{av}\left|G(0)\right|^2=\frac{16}{3}\mathcal{E}_cT_cw_mJ_{av}, \quad \mathcal{E}_c=R_c\mathcal{E}_b$$

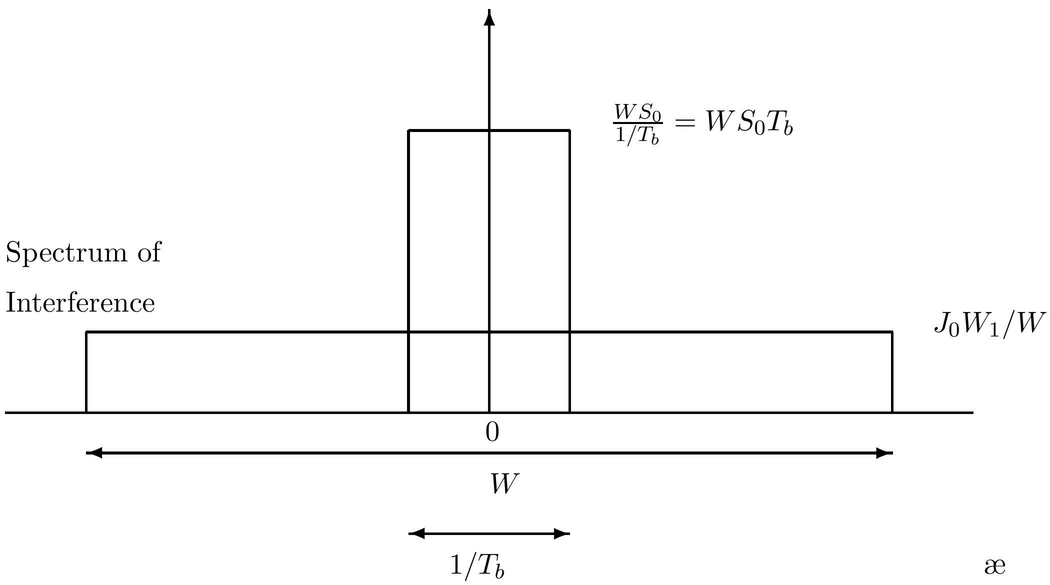
Hence,

$$P_M\leq\sum_{m=2}^{M}Q\left(\sqrt{3\frac{R_c\mathcal{E}_b}{J_{av}T_c}w_m}\right)$$

This is an improvement of 1.76 dB over the rectangular pulse.

Problem 13.2 :

The PN spread spectrum signal has a bandwidth W and the interference has a bandwidth W_1, where $W >> W_1$. Upon multiplication of the received signal $r(t)$ with the PN reference at the receiver, we have the following (approximate) spectral characteristics



After multiplication with the PN reference, the interference power in the bandwidth $1/T_b$ occupied by the signal is

$$\left(\frac{J_0W_1}{W}\right)\left(\frac{1}{T_b}\right) = \frac{J_0W_1}{WT_b}$$

Prior to multiplication, the noise power is J_0W. Therefore, in the bandwidth of the information-bearing signal, there is a reduction in the interference power by a factor $WT_b = \frac{T_b}{T_c} = L_c$, which is just the processing gain of the PN spread spectrum signal.

Problem 13.3 :

The concatenation of a Reed-Solomon (31,3) code with the Hadamard (16,5) code results in an equivalent binary code of black length $n = n_1n_2 = 31 \times 16 = 496$ bits. There are 15 information bits conveyed by each code word, i.e. $k = k_1k_2 = 15$. Hence, the overall code rate is $R_c = 15/496$, which is the product of the two code rates. The minimum distances are

$$\begin{array}{l} \text{Reed} - \text{Solomon code } : D_{\min} = 31 - 3 + 1 = 29 \\ \text{Hadamard code } : d_{\min} = \frac{n_2}{2} = 8 \end{array}$$

Hence, the minimum distance of the overall code is $d_{\min} = 28 \times 8 = 232$. A union bound on the probability of error based on the minimum distance of the code is

$$P_M \leq (M-1)\, Q\left(\sqrt{2\frac{\mathcal{E}_b}{J_{av}T_c}R_c d_{\min}}\right)$$

where $M = 2^{15} = 32768$. Also, $\mathcal{E}_b = S_{av}T_b$. Thus,

$$P_M \leq 2^{15} Q\left(\sqrt{2\frac{S_{av}T_b}{J_{av}T_c}\frac{k}{n}d_{\min}}\right)$$

But $kT_b = nT_c$ and $d_{mn} = 232$. Hence,

$$P_M \leq 2^{15} Q\left(\sqrt{\frac{464}{J_{av}/S_{av}}}\right)$$

Due to the large number of codewords, this union bound is very loose. A much tighter bound is

$$P_M \leq \sum_{m=2}^{M} Q\left(\sqrt{\frac{2w_m}{J_{av}/S_{av}}}\right)$$

but the evaluation of the bound requires the use of the weight distribution of the concatenated code.

Problem 13.4 :

For hard-decision decoding we have

$$P_M \leq (M-1)\left[4p(1-p)\right]^{d_{\min}/2} = 2^{m+1}\left[4p(1-p)\right]^{2^{m-2}}$$

where $p = Q\left(\sqrt{\frac{2W/R}{J_{av}/S_{av}}R_c}\right) = Q\left(\sqrt{2\frac{S_{av}}{J_{av}}}\right)$. Note that in the presence of a strong jammer, the probability p is large, i.e close to 1/2. For soft-decision decoding, the error probability bound is

$$P_M \leq (M-1) Q\left(\sqrt{\frac{2W/R}{J_{av}/S_{av}}R_c d_{\min}}\right)$$

We select $\frac{W}{R} = \frac{n}{k} = \frac{1}{R_c}$ and hence:

$$\begin{aligned} P_M &\leq 2^{m+1} Q\left(\sqrt{\frac{2^m}{J_{av}/S_{av}}}\right) < 2^m \exp\left(-\frac{2^{m-1}}{J_{av}/S_{av}}\right) \\ &< \exp\left(-\frac{S_{av}}{J_{av}}\left[2^{m-1} - \frac{J_{av}}{S_{av}} m \ln 2\right]\right) \end{aligned}$$

For a give jamming margin, we can find an m which is sufficiently large to achieve the desired level of performance.

Problem 13.5 :

(a) The coding gain is

$$R_c d_{\min} = \frac{1}{2} \times 10 = 5\,(7dB)$$

(b) The processing gain is W/R, where $W = 10^7 Hz$ and $R = 2000 bps$. Hence,

$$\frac{W}{R} = \frac{10^7}{2 \times 10^3} = 5 \times 10^3 \, (37dB)$$

(c) The jamming margin is

$$\begin{aligned} \left(\frac{J_{av}}{P_{av}}\right) &= \frac{\left(\frac{W}{R}\right) \cdot (R_c d_m in)}{\left(\frac{\mathcal{E}_b}{J_0}\right)} \Rightarrow \\ \left(\frac{J_{av}}{P_{av}}\right)_{dB} &= \left(\frac{W}{R}\right)_{dB} + (CG)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 37 + 7 - 10 = 34dB \end{aligned}$$

Problem 13.6 :

We assume that the interference is characterized as a zero-mean AWGN process with power spectral density J_0. To achieve an error probability of 10^{-5}, the required $\mathcal{E}_b/J_0 = 10$. Then, by using the relation in (13-2-58) and (13-2-38), we have

$$\begin{aligned} \frac{W/R}{J_{av}/P_{av}} &= \frac{W/R}{N_u - 1} = \frac{\mathcal{E}_b}{J_0} \\ W/R &= \left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \\ W &= R\left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \end{aligned}$$

where $R = 10^4$ bps, $N_u = 30$ and $\mathcal{E}_b/J_0 = 10$. Therefore,

$$W = 2.9 \times 10^6 \; Hz$$

The minimum chip rate is $1/T_c = W = 2.9 \times 10^6$ chips/sec.

Problem 13.7 :

To achieve an error probability of 10^{-6}, we require

$$\left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 10.5dB$$

Then, the number of users of the CDMA system is

$$\begin{aligned} N_u &= \frac{W/R}{\mathcal{E}_b/J_0} + 1 \\ &= \frac{1000}{11.3} + 1 = 89 \text{ users} \end{aligned}$$

If the processing gain is reduced to $W/R = 500$, then

$$N_u = \frac{500}{11.3} + 1 = 45 \text{users}$$

Problem 13.8 :

(a) We are given a system where $(J_{av}/P_{av})_{dB} = 20\ dB, R = 1000\ bps$ and $(\mathcal{E}_b/J_0)_{dB} = 10\ dB$. Hence, using the relation in (13-2-38) we obtain

$$\begin{aligned}
\left(\tfrac{W}{R}\right)_{dB} &= \left(\tfrac{J_{av}}{P_{av}}\right)_{dB} + \left(\tfrac{\mathcal{E}_b}{J_0}\right)_{dB} = 30\ dB \\
\tfrac{W}{R} &= 1000 \\
W &= 1000R = 10^6 Hz
\end{aligned}$$

(b) The duty cycle of a pulse jammer for worst-case jamming is

$$\alpha^* = \frac{0.71}{\mathcal{E}_b/J_0} = \frac{0.7}{10} = 0.07$$

The corresponding probability of error for this worst-case jamming is

$$P_2 = \frac{0.083}{\mathcal{E}_b/J_0} = \frac{0.083}{10} = 8.3 \times 10^{-3}$$

Problem 13.9 :

(a) We have $N_u = 15$ users transmitting at a rate of $10,000\ bps$ each, in a bandwidth of $W = 1\ MHz$. The $\mathcal{E}_b/J_0$ is

$$\begin{aligned}
\tfrac{\mathcal{E}_b}{J_0} &= \tfrac{W/R}{N_u-1} = \tfrac{10^6/10^4}{14} = \tfrac{100}{14} \\
&= 7.14\,(8.54\ dB)
\end{aligned}$$

(b) The processing gain is 100.

(c) With $N_u = 30$ and $\mathcal{E}_b/J_0 = 7.14$, the processing gain should be increased to

$$W/R = (7.14)\,(29) = 207$$

Hence, the bandwidth must be increased to $W = 2.07MHz$.

Problem 13.10 :

The processing gain is given as

$$\frac{W}{R} = 500 \;\; (27 \; dB)$$

The $(\mathcal{E}_b/J_0)$ required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB. Hence, the jamming margin is

$$\begin{aligned} \left(\frac{J_{av}}{P_{av}}\right)_{dB} &= \left(\frac{W}{R}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 27 - 9.5 \\ &= 17.5 \; dB \end{aligned}$$

Problem 13.11 :

If the jammer is a pulse jammer with a duty cycle $\alpha = 0.01$, the probability of error for binary PSK is given as

$$P_2 = \alpha Q\left(\sqrt{\frac{2W/R}{J_{av}/P_{av}}}\right)$$

For $P_2 = 10^{-5}$, and $\alpha = 0.01$, we have

$$Q\left(\sqrt{\frac{2W/R}{J_{av}/P_{av}}}\right) = 10^{-3}$$

Then,

$$\frac{W/R_b}{J_{av}/P_{av}} = \frac{500}{J_{av}/P_{av}} = 5$$

and

$$\frac{J_{av}}{P_{av}} = 100 \;\; (20 \; dB)$$

Problem 13.12 :

$$c(t) = \sum_{n=-\infty}^{\infty} c_n p(t - nT_c)$$

The power spectral density of $c(t)$ is given by (8.1.25) as

$$\Phi_c(f) = \frac{1}{T_c}\Phi_c(f)\left|P(f)\right|^2$$

where

$$\left|P(f)\right|^2 = (AT_c)^2 \sin c^2(fT_c), \quad T_c = 1\mu\sec$$

and $\Phi_c(f)$ is the power spectral density of the sequence $\{c_n\}$. Since the autocorrelation of the sequence $\{c_n\}$ is periodic with period N and is given as

$$\phi_c(m) = \begin{cases} N, & m = 0, \pm N, \pm 2N, \ldots \\ \\ -1, & \text{otherwise} \end{cases}$$

then, $\phi_c(m)$ can be represented in a discrete Fourier series as

$$\phi_c(m) = \frac{1}{N}\sum_{k=0}^{N-1} r_C(k)\, e^{j2\pi mk/N}, m = 0, 1, \ldots, N-1$$

where $\{r_c(k)\}$ are the Fourier series coefficients, which are given as

$$r_c(k) = \sum_{m=0}^{N-1} \phi_c(m)\, e^{-j2\pi km/N}, k = 0, 1, \ldots, N-1$$

and $r_c(k + nN) = r_c(k)$ for $n = 0, \pm 1, \pm 2, \ldots$. The latter can be evaluated to yield

$$\begin{aligned} r_c(k) &= N + 1 - \textstyle\sum_{m=0}^{N-1} e^{-j2\pi km/N} \\ &= \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ N+1, & \text{otherwise} \end{cases} \end{aligned}$$

The power spectral density of the sequence $\{c_n\}$ may be expressed in terms of $\{r_c(k)\}$. These coefficients represent the power in the spectral components at the frequencies $f = k/N$. Therefore, we have

$$\Phi_c(f) = \frac{1}{N}\sum_{k=-\infty}^{\infty} r_c(k)\,\delta\left(f - \frac{k}{NT_c}\right)$$

Finally, we have

$$\Phi_c(f) = \frac{1}{NT_c}\sum_{k=-\infty}^{\infty} r_c(k)\left|P\left(\frac{k}{NT_c}\right)\right|^2 \delta\left(f - \frac{k}{NT_c}\right)$$

Problem 13.13 :

Without loss of generality, let us assume that $N_1 < N_2$. Then, the period of the sequence obtained by forming the modulo-2 sum of the two periodic sequences is

$$N_3 = kN_2$$

where k is the smallest integer multiple of N_2 such that kN_2/N_1 is an integer. For example, suppose that $N_1 = 15$ and $N_2 = 63$. Then, we find the smallest multiple of 63 which is divisible by $N_1 = 15$, without a remainder. Clearly, if we take $k = 5$ periods of N_2, which yields a sequence of $N_3 = 315$, and divide N_3 by N_1, the result is 21. Hence, if we take $21N_1$ and $5N_2$, and modulo-2 add the resulting sequences, we obtain a single period of length $N_3 = 21N_1 = 5N_2$ of the new sequence.

Problem 13.14 :

(a) The period of the maximum length shift register sequence is

$$N = 2^{10} - 1 = 1023$$

Since $T_b = NT_c$, then the processing gain is

$$N\frac{T_b}{T_c} = 1023\,(30dB)$$

(b) According to (132-38 jamming margin is

$$\begin{aligned}\left(\frac{J_{av}}{P_{av}}\right)_{dB} &= \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 30 - 10 \\ &= 20dB\end{aligned}$$

where $J_{av} = J_0 W \approx J_0/T_c = J_0 \times 10^6$

Problem 13.15 :

(a) The length of the shift-register sequence is

$$\begin{aligned}L &= 2^m - 1 = 2^{15} - 1 \\ &= 32767 \text{ bits}\end{aligned}$$

For binary FSK modulation, the minimum frequency separation is $2/T$, where $1/T$ is the symbol (bit) rate. The hop rate is 100 $hops/\sec$. Since the shift register has $L = 32767$ states and each state utilizes a bandwidth of $2/T = 200\ Hz$, then the total bandwidth for the FH signal is 6.5534 MHz.

(b) The processing gain is W/R. We have,

$$\frac{W}{R} = \frac{6.5534 \times 10^6}{100} = 6.5534 \times 10^4 \text{ bps}$$

(c) If the noise is AWG with power spectral density N_0, the probability of error expression is

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{W/R}{P_N/P_{av}}}\right)$$

Problem 13.16 :

(a) If the hopping rate is 2 hops/bit and the bit rate is 100 bits/sec, then, the hop rate is 200 hops/sec. The minimum frequency separation for orthogonality $2/T = 400Hz$. Since there are $N = 32767$ states of the shift register and for each state we select one of two frequencies separated by 400 Hz, the hopping bandwidth is 13.1068 MHz.

(b) The processing gain is W/R, where $W = 13.1068\ MHz$ and $R = 100bps$. Hence

$$\frac{W}{R} = 0.131068\,MHz$$

(c) The probability of error in the presence of AWGN is given by (13-3-2) with $L = 2$ chips per hop.

Problem 13.17 :

(a) The total SNR for three hops is $20 \sim 13$ dB.Therefore the SNR per hop is 20/3. The probability of a chip error with noncoherent detection is

$$p = \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}}$$

where $\mathcal{E}_c/N_0 = 20/3$. The probability of a bit error is

$$\begin{aligned} P_b &= 1 - (1-p)^2 \\ &= 1 - (1 - 2p + p^2) \\ &= 2p - p^2 \\ &= e^{-\frac{\mathcal{E}_c}{2N_0}} - \frac{1}{2}e^{-\frac{\mathcal{E}_c}{N_0}} \\ &= 0.0013 \end{aligned}$$

b) In the case of one hop per bit, the SNR per bit is 20, Hence,

$$\begin{aligned} P_b &= \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}} \\ &= \frac{1}{2}e^{-10} \\ &= 2.27 \times 10^{-5} \end{aligned}$$

THerefore there is a loss in performance of a factor 57 AWGN due to splitting the total signal energy into three chips and, then, using hard decision decoding.

Problem 13.18 :

(a) We are given a hopping bandwidth of 2 GHz and a bit rate of 10 kbs. Hence,

$$\frac{W}{R} = \frac{2 \times 10^9}{10^4} = 2 \times 10^5 \, (53dB)$$

(b) The bandwidth of the worst partial-band jammer is $\alpha^* W$, where

$$\alpha^* = 2/\left(\mathcal{E}_b/J_0\right) = 0.2$$

Hence

$$\alpha^* W = 0.4GHz$$

(c) The probability of error with worst-case partial-band jamming is

$$\begin{aligned} P_2 &= \frac{e^{-1}}{(\mathcal{E}_b/J_0)} = \frac{e^{-1}}{10} \\ &= 3.68 \times 10^{-2} \end{aligned}$$

Problem 13.19 :

The error probability for the binary convolutional code is upper-bounded as :

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

where $P_2(d)$ is the probability of error in a pairwise comparison of two paths that are separated in Hamming distance by d. For square-law detected and combined binary FSK in AWGN, $P_2(d)$ is

$$P_2(d) = \frac{1}{2^{2d-1}} \exp\left(-\gamma_b R_c d/2\right) \sum_{n=0}^{d-1} \left[\frac{1}{n!}\left(\frac{\gamma_b R_c d}{2}\right)^n \sum_{r=0}^{d-1-n} \binom{2d-1}{r}\right]$$

Problem 13.20 :

For hard-decision Viterbi decoding of the convolutional code, the error probability is

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

where $P_2(d)$ is given by (8.2.28) when d is odd and by (8.2.29) when d is even. Alternatively, we may use the Chernoff bound for $P_2(d)$, which is : $P_2(d) \leq [4p(1-p)]^{d/2}$. In both cases, $p = \frac{1}{2}\exp(-\gamma_b R_c/2)$.

Problem 13.21 :

For fast frequency hopping at a rate of L hopes/bit and for soft-decision decoding, the performance of the binary convolutional code is upper bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where

$$P_2(Ld) = \frac{1}{2^{2Ld-1}} \exp\left(-\gamma_b R_c d/2\right) \sum_{n=0}^{Ld-1} \left[\frac{1}{n!}\left(\frac{\gamma_b R_c d}{2}\right)^n \sum_{r=0}^{Ld-1-n} \binom{2Ld-1}{r}\right]$$

Note that the use of L hops/coded bit represents a repetition of each coded bit by a factor of L. Hence, the convolutional code is in cascade with the repetition code. The overall code rate of R_c/L and the distance properties of the convolutional code are multiplied by the factor L, so that the binary event error probabilities are evaluated at distances of Ld, where $d_{free} \leq d \leq \infty$.

Problem 13.22 :

For fast frequency hopping at a rate of L hopes/bit and for hard-decision Viterbi decoding, the performance of the binary convolutional code is upper bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where

$$P_2(d) \leq [4p(1-p)]^{d/2}$$

and

$$p = \frac{1}{2^{2L-1}} e^{-\gamma_b R_c/2} \sum_{n=0}^{L-1} \left[\left(\frac{\gamma_b R_c}{2} \right)^n \frac{1}{n!} \sum_{r=0}^{L-1-n} \binom{2L-1}{r} \right]$$

On the other hand, if each of the L chips is detected independently, then :

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(Ld)$$

where $P_2(Ld) \leq [4p(1-p)]^{Ld/2}$ and p=$\frac{1}{2}\exp(\gamma_b R_c/2)$.

Problem 13.23 :

There are 64 decision variables corresponding to the 64 possible codewords in the (7,2) Reed-Solomon code. With $d_{\min} = 6$, we know that the performance of the code is no worse than the performance of an $M = 64$ orthogonal waveform set, where the SNR per bit is degraded by the factor 6/7. Thus, an upper bound on the code word error probability is

$$P_{64} \leq (M-1)P_2 = 63P_2$$

where

$$P_2 \leq \frac{1}{2^{2L-1}} \exp\left(-\gamma_b \frac{6}{7} k/2\right) \sum_{n=0}^{L-1} c_n \left(\frac{3}{7} k \gamma_b\right)^n$$

With $L = 6$ and $k = 6$, P_2 becomes

$$P_2 \leq \frac{1}{2^{11}} \exp\left(-18\gamma_b/7\right) \sum_{n=0}^{5} c_n \left(\frac{18}{7} \gamma_b\right)^n$$

where

$$c_n = \frac{1}{n!} \sum_{r=0}^{5-n} \binom{11}{r}$$

Problem 13.24 :

In the worst-case partial-band interference channel, the (7,2) Reed-Solomon code provides an effective order of diversity $L = 6$. Hence

$$\begin{aligned} P_{64} &\leq 63P_2(6) = 63 \left(\frac{1.47}{\frac{6}{7}\gamma_b} \right)^6 \\ &\leq \frac{1.6\times10^3}{\gamma_b^6}, \text{ for } \frac{\gamma_b}{L} = \frac{\gamma_b}{6} \geq 3 \end{aligned}$$

Problem 13.25 :

$$P_2(a) = aQ\left(\sqrt{\frac{2a\mathcal{E}_b}{J_0}}\right) = a\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2}dt$$

Hence, the maximum occurs when

$$\frac{dP_2(a)}{da} = \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2}dt - \frac{1}{2}\sqrt{\frac{a\mathcal{E}_b}{\pi J_0}}e^{-\frac{a\mathcal{E}_b}{J_0}} = 0 \Rightarrow$$

$$\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{2a\mathcal{E}_b}{J_0}}}^{+\infty} e^{-t^2/2}dt = \frac{1}{2}\sqrt{\frac{a\mathcal{E}_b}{\pi J_0}}e^{-\frac{a\mathcal{E}_b}{J_0}}$$

The value of a that satisfies the above equation can be found numerically (or graphically) and is equal to $a = \frac{0.71}{\mathcal{E}_b/J_0}$. Substitution of this result into $P_2(a)$ yields (for $\mathcal{E}_b/J_0 \geq 0.71$)

$$P_2 = \frac{0.71}{\mathcal{E}_b/J_0}Q\left(\sqrt{2\cdot 0.71}\right) = \frac{0.083}{\mathcal{E}_b/J_0}$$

Problem 13.26 :

The problem is to determine

$$E\left[\exp\left(-v\sum_{k=1}^{L}\beta_k\left|2\mathcal{E}_c + N_{1k}\right|^2 - v\sum_{k=1}^{L}\beta_k\left|N_{2k}\right|^2\right)\right]$$

where thee $\{\beta_k\}$ are fixed and the $\{N_{1k}\},\{N_{2k}\}$ are complex-valued Gaussian random variables with zero-mean and variances equal to σ_k^2. We note that $\beta_k = 1/\sigma_k^2$ and, hence,

$$\tfrac{1}{2}E\left|N_{1k}\right|^2 = \tfrac{1}{2}E\left|N_{2k}\right|^2 = \sigma_k^2$$

$$\tfrac{1}{2}\beta_k E\left|N_{1k}\right|^2 = \tfrac{1}{2}\beta_k E\left|N_{2k}\right|^2 = 1$$

Since the $\{N_{1k}\}$, and $\{N_{2k}\}$ are all statistically independent

$$E\left[\exp\left(-v\sum_{k=1}^{L}\beta_k\left|2\mathcal{E}_c + N_{1k}\right|^2\right)\right] = \prod_{k=1}^{L}E\left[\exp\left(-v\beta_k\left|2\mathcal{E}_c + N_{1k}\right|^2\right)\right]$$

and similarly for $E\left[\exp\left(-v\sum_{k=1}^{L}\beta_k\left|N_{2k}\right|^2\right)\right]$. Now, we use the characteristic function for the sum of the squares of two Gaussian random variables which is given by (2.1.114) of the text (the two Gaussian random variables are the real and imaginary parts of $2\mathcal{E}_c + N_{1k}$). That is

$$\psi_Y(ju) = E\left(e^{juY}\right) = \frac{1}{1 - ju\sigma^2}\exp\left(\frac{ju\sum_{i=1}^{m}m_i^2}{1 - ju\sigma^2}\right)$$

where $Y = X_1^2 + X_2^2$, m_i is the mean of X_i and σ^2 is the variance of X_i. Hence,

$$E\left[\exp\left(-v\left|\sqrt{\beta_k}2\mathcal{E}_c + \sqrt{\beta_k}N_{1k}\right|^2\right)\right] = \frac{1}{1+2v}\exp\left(\frac{-4\beta_k v\mathcal{E}_c^2}{1+2v}\right)$$

and

$$E\left[\exp\left(-v\left|\sqrt{\beta_k}N_{2k}\right|^2\right)\right] = \frac{1}{1-2v}$$

Consequently

$$P_2(\beta) = \prod_{k=1}^{L}\frac{1}{1-4v^2}\exp\left(\frac{-4\beta_k v\mathcal{E}_c^2}{1+2v}\right)$$

Problem 13.27 :

The function

$$f(v,a) = \left[\frac{a}{1-4v^2}\exp\left(\frac{-2av\mathcal{E}_c^2}{J_0(1+2v)}\right)\right]^L$$

is to be minimized with respect to v and maximized with respect to a. Thus,

$$\frac{\partial}{\partial v}f(v,a) = 0 \Rightarrow 8v(1+2v) - \frac{2a\mathcal{E}_c}{J_0}(1-2v) = 0$$

and

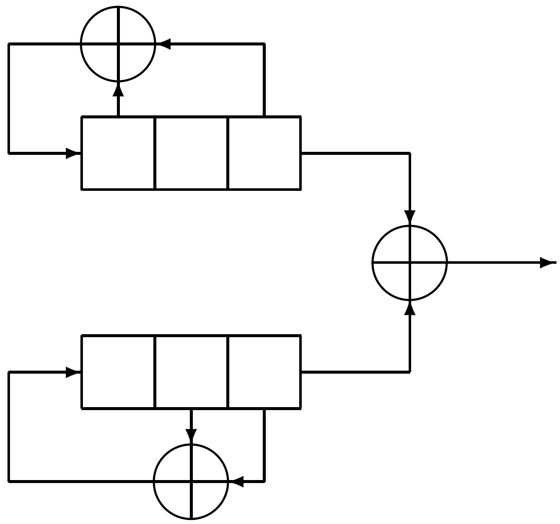
$$\frac{\partial}{\partial a}f(v,a) = 0 \Rightarrow 1 - \frac{2av\mathcal{E}_c}{J_0(1+2v)} = 0$$

The simultaneous solution of these two equations yields the result $v = 1/4$ and $a = \frac{3J_0}{\mathcal{E}_c} \leq 1$. For these values, the function $f(v,a)$ becomes

$$f\left(\frac{1}{4},\frac{3J_0}{\mathcal{E}_c}\right) = \left(\frac{4}{e\gamma_c}\right)^L = \left(\frac{1.47}{\gamma_c}\right)^L \quad \text{for } \gamma_c = \frac{\mathcal{E}_c}{J_0} \geq 3$$

Problem 13.28 :

The Gold code sequences of length $n = 7$ may be generated by the use of two feedback shift registers of length 4 as shown below



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These sequences have the following cross-correlation values

Values of correlation	Frequency of this value
-5	1
3	3
-1	3

Problem 13.29 :

The method of Omura and Levitt (1982) based on the cut-off rate R_0, which is described in Section 13.2.3, can be used to evaluate the error rate of the coded system with interleaving in pulse interference. This computational method yields results that are reasonably close to the results given by Martin and McAdam (1980) and which are illustrated in Fig. 13.2.12 for the rate 1/2 convolutional coder with soft-decision decoding.

Problem 13.30 :

(a) For the coded and interleaved DS binary PSK modulation with pulse jamming and soft-decision decoding, the cutoff rate is

$$R_0 = 1 - \log_2 \left[1 + a e^{-a\mathcal{E}_c/N_0}\right]$$

Hence,

$$\begin{array}{c} \log_2 \left[1 + a e^{-a\mathcal{E}_c/N_0}\right] = 1 - R_0 \Rightarrow \\ \left[1 + a e^{-a\mathcal{E}_c/N_0}\right] = 2^{1-R_0} \Rightarrow \\ \frac{\mathcal{E}_c}{N_0} = \frac{1}{a} \ln \frac{a}{2^{1-R_0}-1} \end{array}$$

and since $\mathcal{E}_c = \mathcal{E}_b R$

$$\frac{\mathcal{E}_b}{N_0} = \frac{1}{aR} \ln \frac{a}{2^{1-R_0} - 1}$$

(b)

$$\frac{d}{da}\left(\frac{\mathcal{E}_b}{N_0}\right) = -\frac{1}{a^2 R} \ln \frac{a}{2^{1-R_0} - 1} + \frac{1}{a^2 R} = 0$$

Hence, the worst-case a is

$$a^* = \left(2^{1-R_0} - 1\right) e$$

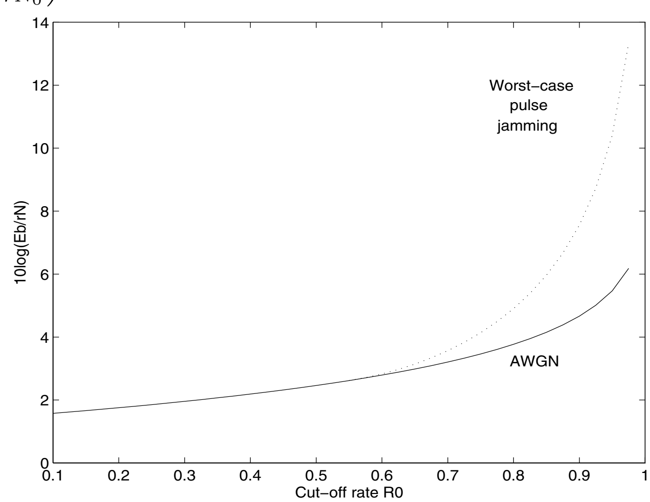
provided that $\left(2^{1-R_0} - 1\right) e < 1$, or equivalently : $R_0 > 1 - \log_2(1 + e^{-1}) = 0.548$. If $R_0 \leq 0.548$, then $a = 1$ maximizes $\frac{\mathcal{E}_b}{N_0}$ and. hence :

$$\frac{\mathcal{E}_b}{N_0} = \frac{1}{R} \ln \frac{1}{2^{1-R_0} - 1}, \quad a = 1$$

which is the result for the AWGN channel. For $R_0 > 0.548$ and $a = a^*$ we have

$$\frac{\mathcal{E}_b}{N_0} = \frac{e^{-1}}{R\,(2^{1-R_0} - 1)}, \quad R_0 > 0.548$$

(c) The plot of $10 \log\left(\frac{\mathcal{E}_b}{rN_0}\right)$ versus R_0, is given in the following figure:



Clearly, there is no penalty in SNR due to worst case pulse jamming for rates below 0.548. Even for $R_0 = 0.8$ the SNR loss is relatively small. As $R_0 \to 1$, the relative difference between pulse jamming and AWGN becomes large.

Problem 13.31 :

(a)

$$R_0 = \log_2 q - \log_2 \left[1 + (q-1)a\exp(-a\mathcal{E}_c/2N_0)\right] \Rightarrow$$
$$1 + (q-1)a\exp(-a\mathcal{E}_c/2N_0) = q2^{-R_0}$$

and since $\mathcal{E}_c = R\mathcal{E}_b$

$$\frac{\mathcal{E}_b}{N_0} = \frac{2}{aR}\ln\frac{(q-1)a}{q2^{-R_0}-1}$$

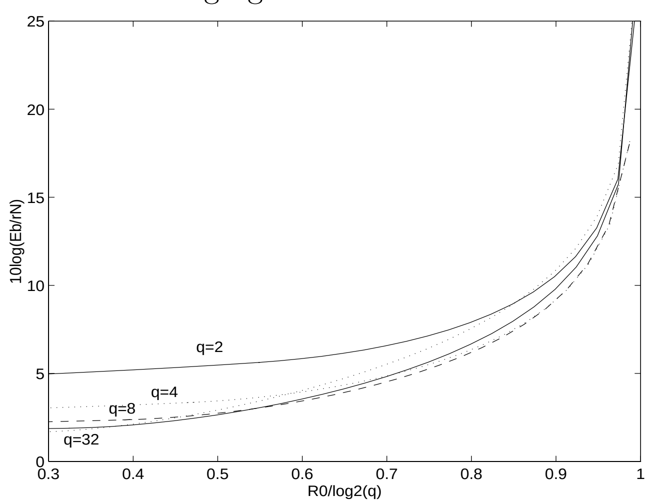
(b)

$$\frac{d}{da}\left(\frac{\mathcal{E}_b}{N_0}\right) = -\frac{2}{a^2R}\ln\frac{(q-1)a}{q2^{-R_0}-1} + \frac{2}{a^2R} = 0 \Rightarrow$$
$$a^* = \frac{\left(q2^{-R_0}-1\right)e}{q-1}$$

provided that $\frac{\left(q2^{-R_0}-1\right)e}{q-1} < 1$ or, equivalently, $R_0 > \log_2\frac{q}{(q-1)/e+1}$. For $a = a^*$, the SNR/bit becomes

$$\frac{\mathcal{E}_b}{N_0} = \frac{2(q-1)}{R\left[q2^{-R_0}-1\right]e}, \quad \text{for } R_0 > \log_2\frac{q}{(q-1)/e+1}$$

(c) The plots are given in the following figure



For low rates, the loss due to partial band jamming is negligible if coding is used. Increasing q reduces the SNR/bit at low rates. At very high rates, a large q implies a large SNR loss. For $q = 2$, there is a 3dB loss relative to binary PSK. As $q \to \infty$, the orthogonal FSK approaches -1.6dB as $R_0 \to 0$.

CHAPTER 14

Problem 14.1 :

Based on the info about the scattering function we know that the multipath spread is $T_m = 1\ ms$, and the Doppler spread is $B_d = 0.2\ Hz$.

(a) (i) $\mathrm{T}_m = 10^{-3}$ sec
(ii) $B_d = 0.2\ Hz$
(iii) $(\Delta t)_c \approx \frac{1}{B_d} = 5$ sec
(iv) $(\Delta f)_c \approx \frac{1}{T_m} = 1000\ Hz$
(v) $\mathrm{T}_m B_d = 2 \cdot 10^{-4}$

(b) (i) Frequency non-selective channel : This means that the signal transmitted over the channel has a bandwidth less that 1000 Hz.
(ii) Slowly fading channel : the signaling interval T is $T << (\Delta t)_c$.
(iii) The channel is frequency selective : the signal transmitted over the channel has a bandwidth greater than 1000 Hz.

(c) The signal design problem does not have a unique solution. We should use orthogonal M=4 FSK with a symbol rate of 50 symbols/sec. Hence $T = 1/50$ sec. For signal orthogonality, we select the frequencies with relative separation $\Delta f = 1/T = 50$ Hz. With this separation we obtain 10000/50=200 frequencies. Since four frequencies are requires to transmit 2 bits, we have up to $50^{th}-$order diversity available. We may use simple repetition-type diversity or a more efficient block or convolutional code of rate $\geq 1/50$. The demodulator may use square-law combining.

Problem 14.2 :

(a)

$$P_{2h} = p^3 + 3p^2(1-p)$$

where $p = \frac{1}{2+\bar{\gamma}_c}$, and $\bar{\gamma}_c$ is the received SNR/cell.

(b) For $\bar{\gamma}_c = 100,\ P_{2h} \approx 10^{-6} + 3 \cdot 10^{-4} \approx 3 \cdot 10^{-4}$
For $\bar{\gamma}_c = 1000,\ P_{2h} \approx 10^{-9} + 3 \cdot 10^{-6} \approx 3 \cdot 10^{-6}$

(c) Since $\bar{\gamma}_c >> 1$, we may use the approximation : $P_{2s} \approx \binom{2L-1}{L}\left(\frac{1}{\bar{\gamma}_c}\right)^L$, where L is the order of diversity. For L=3, we have :

$$P_{2s} \approx \frac{10}{\bar{\gamma}_c^3} \Rightarrow \left\{ \begin{array}{ll} P_{2s} \approx 10^{-5}, & \bar{\gamma}_c = 100 \\ P_{2s} \approx 10^{-8}, & \bar{\gamma}_c = 1000 \end{array} \right\}$$

(d) For hard-decision decoding :

$$P_{2h} = \sum_{k=\frac{L+1}{2}}^{L} \binom{L}{k} p^k (1-p)^{L-k} \leq [4p(1-p)]^{L/2}$$

where the latter is the Chernoff bound, L is odd, and $p = \frac{1}{2+\bar{\gamma}_c}$. For soft-decision decoding :

$$P_{2s} \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c}\right)^L$$

Problem 14.3 :

(a) For a fixed channel, the probability of error is : $P_e(a) = Q\left(\sqrt{\frac{a^2 2\mathcal{E}}{N_0}}\right)$. We now average this conditional error probability over the possible values of α, which are a=0, with probability 0.1, and a=2 with probability 0.9. Thus :

$$P_e = 0.1Q(0) + 0.9Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) = 0.05 + 0.9Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right)$$

(b) As $\frac{\mathcal{E}}{N_0} \rightarrow \infty, \ P_e \rightarrow 0.05$

(c) When the channel gains a_1, a_2 are fixed, the probability of error is :

$$P_e(a_1, a_2) = Q\left(\sqrt{\frac{(a_1^2 + a_2^2)\, 2\mathcal{E}}{N_0}}\right)$$

Averaging over the probability density function $p(a_1, a_2) = p(a_1) \cdot p(a_2)$, we obtain the average probability of error :

$$\begin{array}{rl} P_e = & (0.1)^2 Q(0) + 2 \cdot 0.9 \cdot 0.1 \cdot Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + (0.9)^2 Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \\ = & 0.005 + 0.18Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + 0.81Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right) \end{array}$$

(d) As $\frac{\mathcal{E}}{N_0} \rightarrow \infty, \ P_e \rightarrow 0.005$

Problem 14.4 :

(a)

$$T_m = 1 \text{ sec} \Rightarrow (\Delta f)_c \approx \frac{1}{T_m} = 1\ Hz$$
$$B_d = 0.01\ Hz \Rightarrow (\Delta t)_c \approx \frac{1}{B_d} = 100 \text{ sec}$$

(b) Since $W = 5\ Hz$ and $(\Delta f)_c \approx 1\ Hz$, the channel is frequency selective.

(c) Since T=10 sec $<\ (\Delta t)_c$, the channel is slowly fading.

(d) The desired data rate is not specified in this problem, and must be assumed. Note that with a pulse duration of $T = 10$ sec, the binary PSK signals can be spaced at $1/T = 0.1\ Hz$ apart. With a bandwidth of W=5 Hz, we can form 50 subchannels or carrier frequencies. On the other hand, the amount of diversity available in the channel is $W/(\Delta f)_c = 5$. Suppose the desired data rate is 1 bit/sec. Then, ten adjacent carriers can be used to transmit the data in parallel and the information is repeated five times using the total number of 50 subcarriers to achieve 5-th order diversity. A subcarrier separation of 1 Hz is maintained to achieve independent fading of subcarriers carrying the same information.

(e) We use the approximation :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{4\bar{\gamma}_c}\right)^L$$

where L=5. For $P_3 = 10^{-6}$, the SNR required is :

$$(126) \left(\frac{1}{4\bar{\gamma}_c}\right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 10.4 \ (10.1\ dB)$$

(f) The tap spacing between adjacent taps is 1/5=0.2 seconds. the total multipath spread is $T_m = 1$ sec . Hence, we employ a RAKE receiver with at least 5 taps.

(g) Since the fading is slow relative to the pulse duration, in principle we can employ a coherent receiver with pre-detection combining.

(h) For an error rate of 10^{-6}, we have :

$$P_2 \approx \binom{2L-1}{L} \left(\frac{1}{\bar{\gamma}_c}\right)^5 = 10^{-6} \Rightarrow \bar{\gamma}_c = 41.6 \ (16.1\ dB)$$

Problem 14.5 :

(a)

$$p\left(n_1,n_2\right)=\frac{1}{2\pi\sigma^2}e^{-\left(n_1^2+n_2^2\right)/2\sigma^2}$$
$$U_1=2\mathcal{E}+N_1,\ U_2=N_1+N_2\Rightarrow N_1=U_1-2\mathcal{E},\ N_2=U_2-U_1+2\mathcal{E}$$

where we assume that $s(t)$ was transmitted. Then, the Jacobian of the transformation is :

$$J=\left|\begin{array}{cc}1 & -1\\ 0 & 1\end{array}\right|=1$$

and :

$$\begin{aligned}p(u_1,u_2) &= \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left[(U_1-2\mathcal{E})^2+(U_2-(U_1-2\mathcal{E}))^2\right]}\\ &= \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}\left[(U_1-2\mathcal{E})^2+U_2^2+(U_1-2\mathcal{E})^2-2U_2(U_1-2\mathcal{E})\right]}\\ &= \frac{1}{2\pi\sigma^2}e^{-\frac{1}{\sigma^2}\left[(U_1-2\mathcal{E})^2+\frac{1}{2}U_2^2-U_2(U_1-2\mathcal{E})\right]}\end{aligned}$$

The derivation is exactly the same for the case when $-s(t)$ is transmitted, with the sole difference that $U_1=-2\mathcal{E}+N_1$.

(b) The likelihood ratio is :

$$\Lambda=\frac{p(u_1,u_2|+s(t))}{p(u_1,u_2|-s(t))}=\exp\left[-\frac{1}{\sigma^2}\left(-8\mathcal{E}U_1+4\mathcal{E}U_2\right)\right]>^{+s(t)}1$$

or :

$$\ln\Lambda=\frac{8\mathcal{E}}{\sigma^2}\left(U_1-\frac{1}{2}U_2\right)>^{+s(t)}0\Rightarrow U_1-\frac{1}{2}U_2>^{+s(t)}0$$

Hence $\beta=-1/2$.

(c)

$$U=U_1-\frac{1}{2}U_2=2\mathcal{E}+\frac{1}{2}\left(N_1-N_2\right)$$
$$E\left[U\right]=2\mathcal{E},\ \sigma_U^2=\frac{1}{4}\left(\sigma_{n1}^2+\sigma_{n2}^2\right)=\mathcal{E}N_0$$

Hence:

$$p(u)=\frac{1}{\sqrt{2\pi\mathcal{E}N_0}}e^{-(u-2\mathcal{E})^2/2\mathcal{E}N_0}$$

(d)

$$\begin{aligned}P_e &= P(U<0)\\ &= \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi\mathcal{E}N_0}}e^{-(u-2\mathcal{E})^2/2\mathcal{E}N_0}du\\ &= Q\left(\frac{2\mathcal{E}}{\sqrt{\mathcal{E}N_0}}\right)=Q\left(\sqrt{\frac{4\mathcal{E}}{N_0}}\right)\end{aligned}$$

(e) If only U_1 is used in reaching a decision, then we have the usual binary PSK probability of error : $P_e=Q\left(\sqrt{\frac{2E}{N_0}}\right)$, hence a loss of 3 dB, relative to the optimum combiner.

Problem 14.6:

(a)

$$U = Re\left[\sum_{k=1}^{L} \beta_k U_k\right] >^1 0$$

where $U_k = 2Ea_k e^{-j\phi_k} + v_k$ and where v_k is zero-mean Gaussian with variance $2EN_{0k}$. Hence, U is Gaussian with :

$$\begin{aligned} E[U] &= Re\left[\sum_{k=1}^{L} \beta_k E(U_k)\right] \\ &= 2\mathcal{E} \cdot Re\left[\sum_{k=1}^{L} a_k \beta_k e^{-j\phi_k}\right] \\ &= 2\mathcal{E} \sum_{k=1}^{L} a_k |\beta_k| \cos(\theta_k - \phi_k) \equiv m_u \end{aligned}$$

where $\beta_k = |\beta_k| e^{j\theta_k}$. Also :

$$\sigma_u^2 = 2\mathcal{E} \sum_{k=1}^{L} |\beta_k|^2 N_{0k}$$

Hence :

$$p(u) = \frac{1}{\sqrt{2\pi}\sigma_u} e^{-(u-m_u)^2/2\sigma_u^2}$$

(b) The probability of error is :

$$P_2 = \int_{-\infty}^{0} p(u)du = Q\left(\sqrt{2\gamma}\right)$$

where :

$$\gamma = \frac{\mathcal{E}\left[\sum_{k=1}^{L} a_k |\beta_k| \cos(\theta_k - \phi_k)\right]^2}{\sum_{k=1}^{L} |\beta_k|^2 N_{0k}}$$

(c) To maximize P_2, we maximize γ. It is clear that γ is maximized with respect to $\{\theta_k\}$ by selecting $\theta_k = \phi_k$ for $k = 1, 2, ..., L$. Then we have :

$$\gamma = \frac{\mathcal{E}\left[\sum_{k=1}^{L} a_k |\beta_k|\right]^2}{\sum_{k=1}^{L} |\beta_k|^2 N_{0k}}$$

Now :

$$\frac{d\gamma}{d|\beta_l|} = 0 \Rightarrow \left(\sum_{k=1}^{L} |\beta_k|^2 N_{0k}\right) a_l - \left(\sum_{k=1}^{L} a_k |\beta_k|\right) |\beta_l| N_{ol} = 0$$

Consequently :

$$|\beta_l| = \frac{a_l}{N_{0l}}$$

and :

$$\gamma = \frac{\mathcal{E}\left[\sum_{k=1}^{L}\frac{a_k^2}{N_{ok}}\right]^2}{\sum_{k=1}^{L}\frac{a_k^2}{N_{ok}^2}N_{0k}} = \mathcal{E}\sum_{k=1}^{L}\frac{a_k^2}{N_{ok}}$$

The above represents maximal ratio combining.

Problem 14.7 :

(a)

$$p\left(u_1\right) = \frac{1}{\left(2\sigma_1^2\right)^L (L-1)!}u_1^{L-1}e^{-u_1/2\sigma_1^2},\ \sigma_1^2 = 2\mathcal{E}N_0\left(1+\bar{\gamma}_c\right)$$

$$p\left(u_2\right) = \frac{1}{\left(2\sigma_2^2\right)^L (L-1)!}u_2^{L-1}e^{-u_2/2\sigma_2^2},\ \sigma_2^2 = 2\mathcal{E}N_0$$

$$P_2 = P(U_2 > U_1) = \int_0^\infty P(U_2 > U_1|U_1)p(U_1)dU_1$$

But :

$$\begin{aligned} P(U_2 > U_1|U_1) &= \int_{u_1}^\infty p(u_2)du_2 = \int_{u_1}^\infty \frac{1}{\left(2\sigma_2^2\right)^L (L-1)!}u_2^{L-1}e^{-u_2/2\sigma_2^2}du_2 \\ &= \left[\frac{1}{\left(2\sigma_2^2\right)^L (L-1)!}u_2^{L-1}e^{-u_2/2\sigma_2^2}(-2\sigma_2^2)\right]_{u_1}^\infty - \int_{u_1}^\infty \frac{(-2\sigma_2^2)(L-1)}{\left(2\sigma_2^2\right)^L (L-1)!}u_2^{L-2}e^{-u_2/2\sigma_2^2}du_2 \\ &= \frac{1}{\left(2\sigma_2^2\right)^{L-1} (L-1)!}u_1^{L-1}e^{-u_1/2\sigma_2^2} + \int_{u_1}^\infty \frac{1}{\left(2\sigma_2^2\right)^{L-1} (L-2)!}u_2^{L-2}e^{-u_2/2\sigma_2^2}du_2 \end{aligned}$$

Continuing, in the same way, the integration by parts, we obtain :

$$P(U_2 > U_1|U_1) = e^{-u_1/2\sigma_2^2}\sum_{k=0}^{L-1}\frac{\left(u_1/2\sigma_2^2\right)^k}{k!}$$

Then :

$$\begin{aligned} P_2 &= \int_0^\infty \left[e^{-u_1/2\sigma_2^2}\sum_{k=0}^{L-1}\frac{\left(u_1/2\sigma_2^2\right)^k}{k!}\right]\frac{1}{\left(2\sigma_1^2\right)^L (L-1)!}u_1^{L-1}e^{-u_1/2\sigma_1^2}du_1 \\ &= \sum_{k=0}^{L-1}\frac{1}{k!\left(2\sigma_2^2\right)^k\left(2\sigma_1^2\right)^L (L-1)!}\int_0^\infty u_1^{L-1+k}e^{-u_1(1/\sigma_1^2+1/\sigma_2^2)/2}du_1 \end{aligned}$$

The integral that exists inside the summation is equal to :

$$\int_0^\infty u^{L-1+k}e^{-ua}du =$$

$$\left[\frac{u^{L-1+k}e^{-ua}}{(-a)}\right]_0^\infty - \frac{L-1+k}{(-a)}\int_0^\infty u^{L-2+k}e^{-ua}du =$$

$$\frac{L-1+k}{a}\int_0^\infty u^{L-2+k}e^{-ua}du$$

where $a = (1/\sigma_1^2 + 1/\sigma_2^2)/2 = \frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}$. Continuing the integration by parts, we obtain :

$$\int_0^\infty u^{L-1+k} e^{-ua} du = \frac{1}{a^{L+k}} (L-1+k)! = \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^{L+k} (L-1+k)!$$

Hence :

$$\begin{aligned}
P_2 &= \sum_{k=0}^{L-1} \frac{1}{k!\left(2\sigma_2^2\right)^k \left(2\sigma_1^2\right)^L (L-1)!} \int_0^\infty u_1^{L-1+k} e^{-u_1(1/\sigma_1^2+1/\sigma_2^2)/2} du_1 \\
&= \sum_{k=0}^{L-1} \frac{1}{k!\left(2\sigma_2^2\right)^k \left(2\sigma_1^2\right)^L (L-1)!} \left(\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^{L+k} (L-1+k)! \\
&= \sum_{k=0}^{L-1} \binom{L-1+k}{k} \frac{\sigma_1^{2k}\sigma_2^{2L}}{\left(\sigma_1^2+\sigma_2^2\right)^{L+k}} = \sum_{k=0}^{L-1} \binom{L-1+k}{k} \frac{(2EN_0(1+\bar{\gamma}_c))^k (2EN_0)^L}{(2EN_0(1+\bar{\gamma}_c)+2EN_0)^{L+k}} \\
&= \sum_{k=0}^{L-1} \binom{L-1+k}{k} \frac{(2EN_0(1+\bar{\gamma}_c))^k (2EN_0)^L}{(2EN_0(2+\bar{\gamma}_c))^{L+k}} = \left(\frac{1}{2+\bar{\gamma}_c}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\bar{\gamma}_c}{2+\bar{\gamma}_c}\right)^k
\end{aligned}$$

which is the desired expression (14-4-15) with $\mu = \frac{\bar{\gamma}_c}{2+\bar{\gamma}_c}$.

Problem 14.8 :

$$T(D,N,J=1) = \frac{J^3ND^6}{1-JND^2(1+J)}|_{J=1} = \frac{ND^6}{1-2ND^2}$$

$$\frac{dT(D,N)}{dN}|_{N=1} = \frac{\left(1-2ND^2\right)D^6 - ND^6\left(-2D^2\right)}{\left(1-2ND^2\right)^2}|_{N=1} = \frac{D^6}{\left(1-2D^2\right)^2}$$

(a) For hard-decision decoding :

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d) = \frac{dT(D,N)}{dN}|_{N=1, D=\sqrt{4p(1-p)}}$$

where $p = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}}\right]$ for coherent PSK (14-3-7). Thus :

$$P_b \leq \frac{\left[4p(1-p)\right]^3}{\left[1-8p(1-p)\right]^2}$$

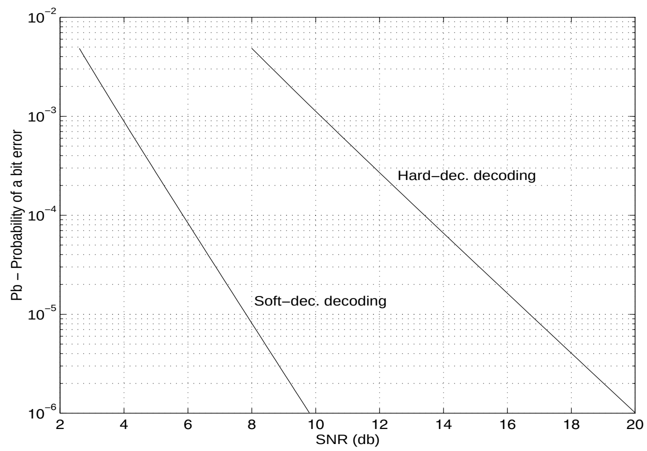
(b) For soft-decision decoding, the error probability is upper bounded as

$$P_b \leq \sum_{d=d_{free}}^{\infty} \beta_d P_2(d)$$

where $d_{free} = 6$, $\{\beta_d\}$ are the coefficients in the expansion of the derivative of T(D,N) evaluated at N=1, and :

$$P_2(d) = \left(\frac{1-\mu}{2}\right)^d \sum_{k=0}^{d-1} \binom{d-1+k}{k} \left(\frac{1+\mu}{2}\right)^k$$

where $\mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}}$, as obtained from (14-4-15).
These probabilities P_b are plotted on the following graph, with $SNR = \bar{\gamma}_c$.



Problem 14.9 :

$$U = \sum_{k=1}^{L} U_k$$

(a) $U_k = 2Ea_k + v_k$, where v_k is Gaussian with $E[v_k] = 0$ and $\sigma_v^2 = 2EN_0$. Hence, for fixed $\{a_k\}$, U is also Gaussian with : $E[U] = \sum_{k=1}^{L} E(U_k) = 2E\sum_{k=1}^{L} a_k$ and $\sigma_u^2 = L\sigma_v^2 = 2LEN_0$. Since U is Gaussian, the probability of error, conditioned on a fixed number of gains $\{a_k\}$ is

$$P_b(a_1, a_2, ..., a_L) = Q\left(\frac{2E\sum_{k=1}^{L} a_k}{\sqrt{2LEN_0}}\right) = Q\left(\sqrt{\frac{2E\left(\sum_{k=1}^{L} a_k\right)^2}{LN_0}}\right)$$

(b) The average probability of error for the fading channel is the conditional error probability averaged over the $\{a_k\}$. Hence :

$$P_d = \int_0^\infty da_1 \int_0^\infty da_2 ... \int_0^\infty da_L P_b(a_1, a_2, ..., a_L)\, p(a_1)p(a_2)...p(a_L)$$

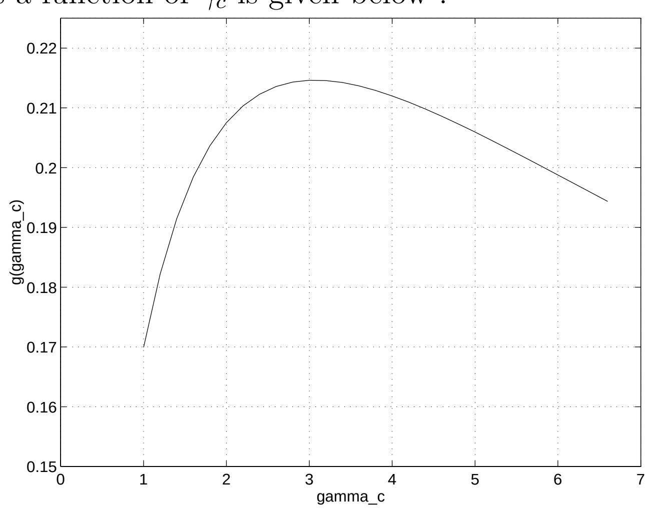
where $p(a_k) = \frac{a_k}{\sigma^2}\exp(-a_k^2/2\sigma^2)$, where σ^2 is the variance of the Gaussian RV's associated with the Rayleigh distribution of the $\{a_k\}$ (not to be confused with the variance of the noise terms). Since $P_b(a_1, a_2, ..., a_L)$ depends on the $\{a_k\}$ through their sum, we may let : $X = \sum_{k=1}^{L} a_k$ and, thus, we have the conditional error probability $P_b(X) = Q\left(\sqrt{2EX/(LN_0)}\right)$. The average error probability is :

$$P_b = \int_0^\infty P_b(X)p(X)dX$$

The problem is to determine $p(X)$. Unfortunately, there is no closed form expression for the pdf of a sum of Rayleigh distributed RV's. Therefore, we cannot proceed any further.

Problem 14.10 :

(a) The plot of $g(\bar{\gamma}_c)$ as a function of $\bar{\gamma}_c$ is given below :



The maximum value of $g(\bar{\gamma}_c)$ is approximately 0.215 and occurs when $\bar{\gamma}_c \approx 3$.

(b) $\bar{\gamma}_c = \bar{\gamma}_b/L$. Hence, for a given $\bar{\gamma}_b$ the optimum diversity is $L = \bar{\gamma}_b/\bar{\gamma}_c = \bar{\gamma}_b/3$.

(c) For the optimum diversity we have :

$$P_2(L_{opt}) < 2^{-0.215\bar{\gamma}_b} = e^{-\ln 2 \cdot 0.215\bar{\gamma}_b} = e^{-0.15\bar{\gamma}_b} = \frac{1}{2}e^{-0.15\bar{\gamma}_b + \ln 2}$$

For the non-fading channel :$P_2 = \frac{1}{2}e^{-0.5\gamma_b}$. Hence, for large SNR ($\bar{\gamma}_b >> 1$), the penalty in SNR is:

$$10\log_{10}\frac{0.5}{0.15} = 5.3\ dB$$

Problem 14.11 :

The radio signal propagates at the speed of light, $c = 3 \times 10^8 m/\sec$. The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3 \times 10^8} = 1\mu \sec$$

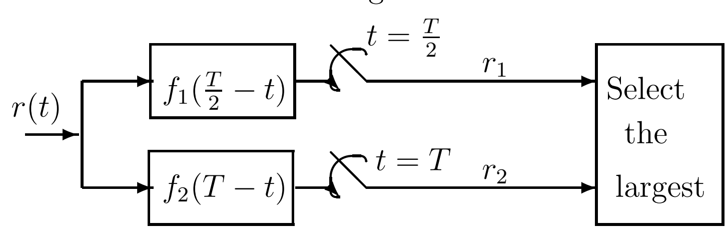
The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is $W = 1\ MHz$. Hence, the minimum chip rate is 10^6 chips per second.

Problem 14.12 :

(a) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \qquad f_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

(b) The optimal receiver is shown in the next figure



(c) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\mathbf{r} = [\sqrt{\frac{A^2T}{2}} + n_1, n_2]$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(e|s_1)$ is

$$\begin{aligned} P(e|s_1) &= P(n_2 - n_1 > \sqrt{\frac{A^2T}{2}}) \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_{\frac{A^2T}{2}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

where we have used the fact the $n = n_2 - n_1$ is a zero-mean Gaussian random variable with variance N_0. Similarly we find that $P(e|s_1) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$, so that

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$$

(d) The signal waveform $f_1(\frac{T}{2}-t)$ matched to $f_1(t)$ is exactly the same with the signal waveform $f_2(T-t)$ matched to $f_2(t)$. That is,

$$f_1(\frac{T}{2}-t) = f_2(T-t) = f_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce the random variables r_1 and r_2 respectively.

(e) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_1 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_1 \\ r_2 &= \frac{A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 = \frac{1}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} V$$

then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned}
r_1 &= n_1 \\
r_2 &= A\sqrt{\frac{2}{T}}\frac{T}{4} + \frac{3A}{2}\sqrt{\frac{2}{T}}\frac{T}{4} + n_2 \\
&= \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2
\end{aligned}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$$

Thus, the average probability of error is given by

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\
&= \frac{1}{2}Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]
\end{aligned}$$

The optimal value of V can be found by setting $\frac{\vartheta P(e)}{\vartheta V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta V} = 0 = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$$

or by solving in terms of V

$$V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

(f) Let a be fixed to some value between 0 and 1. Then, if we argue as in part (e) we obtain

$$\begin{aligned}
P(e|s_1, a) &= P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V(a)) \\
P(e|s_2, a) &= P(n_1 - n_2 > (a+2)\sqrt{\frac{A^2T}{8}} + V(a))
\end{aligned}$$

and the probability of error is

$$P(e|a) = \frac{1}{2}P(e|s_1, a) + \frac{1}{2}P(e|s_2, a)$$

For a given a, the optimal value of $V(a)$ is found by setting $\frac{\vartheta P(e|a)}{\vartheta V(a)}$ equal to zero. By doing so we find that

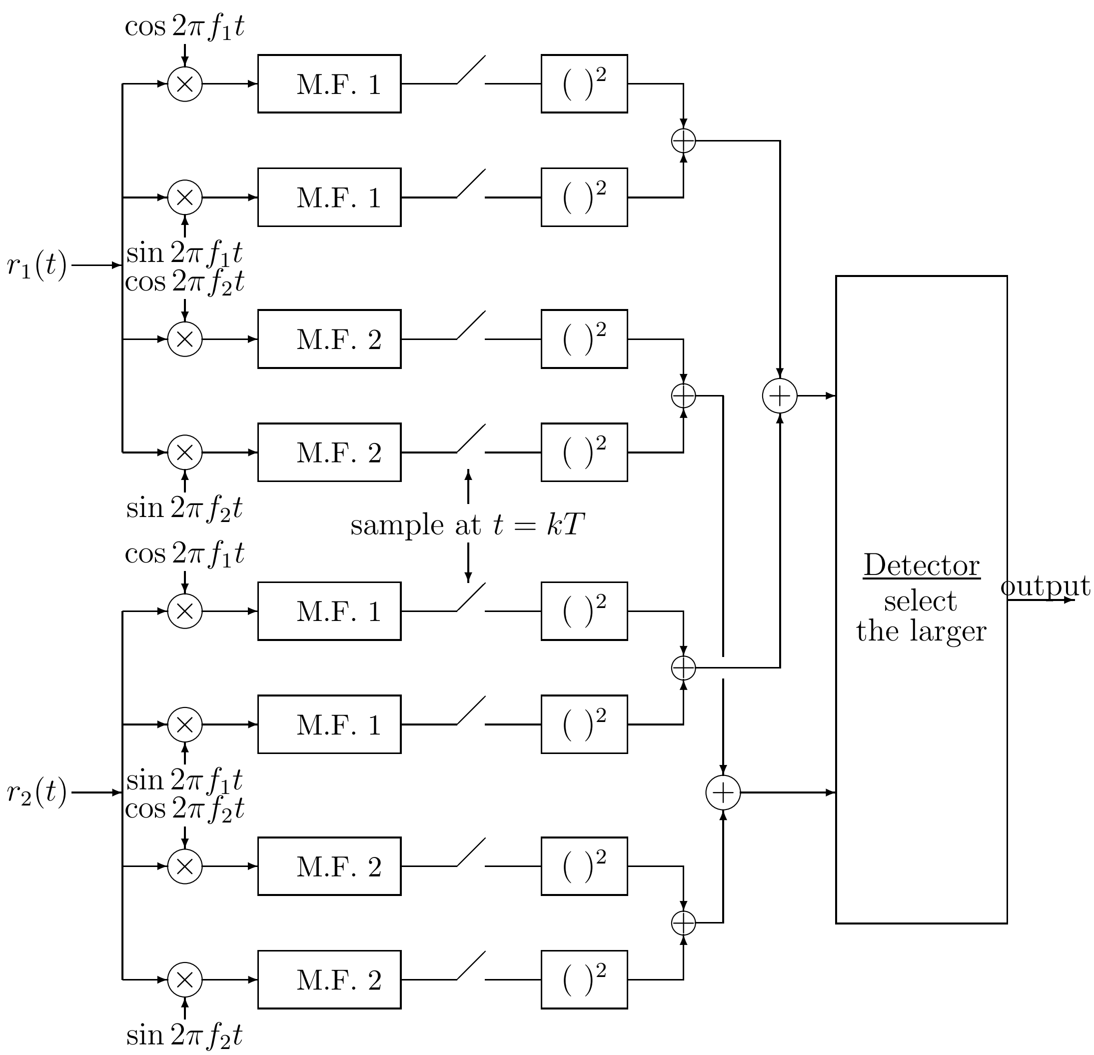
$$V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$$

The mean square estimation of $V(a)$ is

$$V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}}\int_0^1 ada = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

Problem 14.13 :

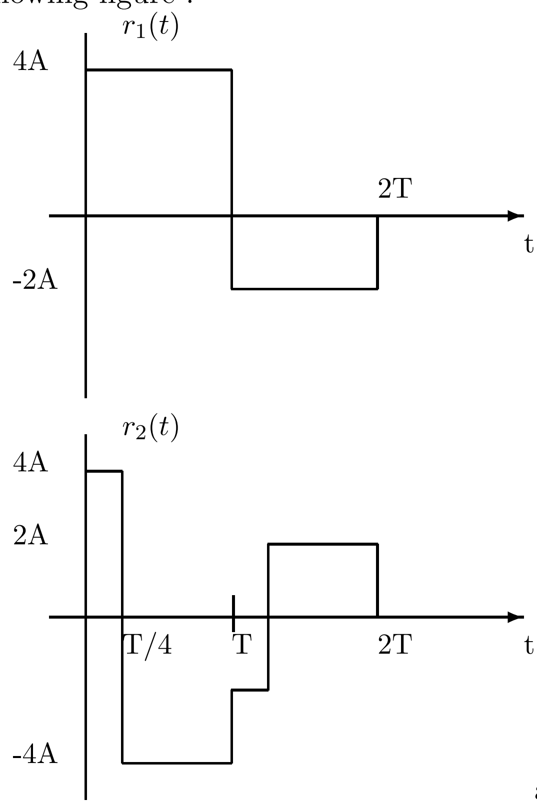
(a)



(b) The probability of error for binary FSK with square-law combining for $L = 2$ is given in Figure 14-4-7. The probability of error for $L = 1$ is also given in Figure 14-4-7. Note that an increase in SNR by a factor of 10 reduces the error probability by a factor of 10 when $L = 1$ and by a factor of 100 when $D = 2$.

Problem 14.14 :

(a) The noise-free received waveforms $\{r_i(t)\}$ are given by : $r_i(t) = h(t) * s_i(t),\ i = 1, 2$, and they are shown in the following figure :



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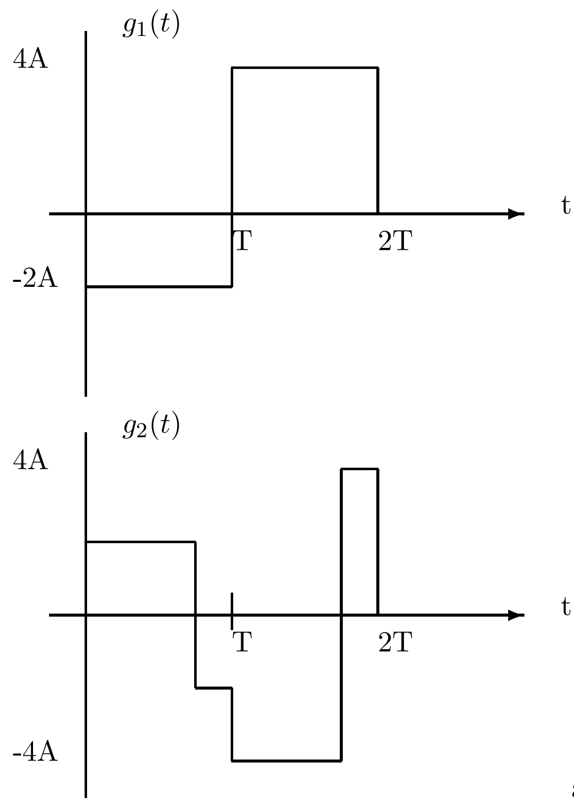
(b) The optimum receiver employs two matched filters $g_i(t) = r_i(2T - t)$, and after each matched filter there is a sampler working at a rate of $1/2T$. The equivalent lowpass responses $g_i(t)$ of the two matched filters are given in the following figure :



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Problem 14.15 :

Since a follows the Nakagami-m distribution :

$$p_a(a) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m a^{2m-1} \exp\left(-ma^2/\Omega\right), \quad a \geq 0$$

where : $\Omega = E\left(a^2\right)$. The pdf of the random variable $\gamma = a^2 \mathcal{E}_b/N_0$ is specified using the usual method for a function of a random variable :

$$a = \sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}, \quad \frac{d\gamma}{da} = 2a\mathcal{E}_b/N_0 = 2\sqrt{\gamma \mathcal{E}_b/N_0}$$

Hence :

$$\begin{aligned} p_\gamma(\gamma) &= \left(\frac{d\gamma}{da}\right)^{-1} p_a\left(\sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}\right) \\ &= \frac{1}{2\sqrt{\gamma \mathcal{E}_b/N_0}} \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m \left(\sqrt{\gamma \frac{N_0}{\mathcal{E}_b}}\right)^{2m-1} \exp\left(-m\gamma \frac{N_0}{\mathcal{E}_b}/\Omega\right) \\ &= \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\Omega^m (\mathcal{E}_b/N_0)^m} \exp\left(-m\gamma/\left(\mathcal{E}_b\Omega/N_0\right)\right) \\ &= \frac{m^m}{\Gamma(m)} \frac{\gamma^{m-1}}{\bar{\gamma}^m} \exp\left(-m\gamma/\bar{\gamma}\right) \end{aligned}$$

where $\bar{\gamma} = E\left(a^2\right)\mathcal{E}_b/N_0$.

Problem 14.16 :

(a) By taking the conjugate of $r_2 = h_1 s_2^* - h_2 s_1^* + n_2$

$$\begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} = \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2^* \end{bmatrix}$$

Hence, the soft-decision estimates of the transmitted symbols (s_1, s_2) will be

$$\begin{aligned} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} &= \begin{bmatrix} h_1 & h_2 \\ -h_2^* & h_1^* \end{bmatrix}^{-1} \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} \\ &= \frac{1}{h_1^2+h_2^2} \begin{bmatrix} h_1^* r_1 - h_2 r_2^* \\ h_2^* r_1 + h_1 r_2^* \end{bmatrix} \end{aligned}$$

which corresponds to dual-diversity reception for s_i.

(b) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-15), with $L = 2$ and $\mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}}$ (where the average SNR per channel is $\bar{\gamma}_c = \frac{\mathcal{E}}{N_0}E[h^2] = \frac{\mathcal{E}}{N_0}$) Then (14.4-15) becomes

$$\begin{aligned} P_2 &= \left[\tfrac{1}{2}(1-\mu)\right]^2 \left\{1\binom{1}{0} + \left[\tfrac{1}{2}(1+\mu)\right]\binom{2}{1}\right\} \\ &= \left[\tfrac{1}{2}(1-\mu)\right]^2 [2+\mu] \end{aligned}$$

When $\bar{\gamma}_c >> 1$, then $\frac{1}{2}(1-\mu) \approx 1/4\bar{\gamma}_c$ and $\mu \approx 1$. Hence, for large SNR the bit error probability for binary PSK can be approximated as

$$P_2 \approx 3\left(\frac{1}{4\bar{\gamma}_c}\right)^2$$

(c) The bit error probability for dual diversity reception of binary PSK is given by Equation (14.4-41), with $L = 2$ and μ as above. Replacing we get

$$P_2 = \frac{1}{2}\left[1 - \frac{\mu}{\sqrt{2-\mu^2}}\left(1 + \frac{1-\mu^2}{2-\mu^2}\right)\right]$$

Problem 14.17 :

(a) Noting that $\mu < 1$, the expression (14.6-35) for the binary event error probability can be upperbounded by

$$\begin{aligned} P_2(w_m) &< \left(\frac{1-\mu}{2}\right)^{w_m} \sum_{k=0}^{w_m-1} \left(\binom{w_m-1+k}{k}\right) \\ &= \left(\frac{1-\mu}{2}\right)^{w_m} \binom{2w_m-1}{w_m} \end{aligned}$$

Hence, the union bound for the probability for a code word error would be:

$$\begin{aligned} P_M &< (M-1)P_2(d_{min}) \\ &< 2^k \binom{2d_{min}-1}{d_{min}} \left(\frac{1-\mu}{2}\right)^{d_{min}} \end{aligned}$$

Now, taking the expression for μ for each of the three modulation schemes, we obtain the desired expression.

Non-coherent FSK :

$$\mu = \frac{\bar{\gamma}_c}{2+\bar{\gamma}_c} \Rightarrow \frac{1-\mu}{2} = \frac{1}{2+\bar{\gamma}_c} < \frac{1}{\bar{\gamma}_c} = \frac{1}{R_C\bar{\gamma}_b}$$

DPSK :

$$\mu = \frac{\bar{\gamma}_c}{1+\bar{\gamma}_c} \Rightarrow \frac{1-\mu}{2} = \frac{1}{2(1+\bar{\gamma}_c)} < \frac{1}{2\bar{\gamma}_c} = \frac{1}{2R_C\bar{\gamma}_b}$$

BPSK : $\mu = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} = \sqrt{1-\frac{1}{1+\bar{\gamma}_c}}$. Using Taylor's expansion, we can approximate for large $\bar{\gamma}_c$ $(1-x)^{1/2} \approx 1 - x/2$. Hence

$$\frac{1-\mu}{2} \approx \frac{1}{4(1+\bar{\gamma}_c)} < \frac{1}{4\bar{\gamma}_c} = \frac{1}{4R_C\bar{\gamma}_b}$$

(b) Noting that

$$\begin{aligned} \exp\left(-d_{min}R_c\bar{\gamma}_b f(\bar{\gamma}_c)\right) &= \exp\left(-d_{min}R_c\bar{\gamma}_b \ln(\beta\bar{\gamma}_c)/\bar{\gamma}_c\right) \\ &= \exp\left(-d_{min}\ln(\beta\bar{\gamma}_c)\right) \\ &= \left(\frac{1}{\beta\bar{\gamma}_c}\right)^{d_{min}} = \left(\frac{1}{\beta R_c\bar{\gamma}_b}\right)^{d_{min}} \end{aligned}$$

we show the equivalence between the expressions of (a) and (b).
The maximum is obtained with

$$\frac{d}{d\bar{\gamma}_c}f(\gamma) = 0 \Rightarrow \frac{\beta}{\beta\bar{\gamma}_c\bar{\gamma}_c} - \frac{\ln(\beta\bar{\gamma}_c)}{\gamma^2} = 0 \Rightarrow \ln(\beta\bar{\gamma}_c) = 1 \Rightarrow \bar{\gamma}_c = \frac{e}{\beta}$$

By checking the second derivative, we verify that this extreme point is indeed a maximum.

(c) For the value of $\bar{\gamma}_c$ found in (b), we have $f_{max}(\bar{\gamma}_c) = \beta/e$. Then

$$\begin{aligned} \exp\left(-k(\beta d_{min}\bar{\gamma}_b/ne - \ln 2)\right) &= \exp(k\ln 2)\exp\left(-R_c\beta d_{min}\bar{\gamma}_b/e\right) \\ &= \exp(k\ln 2)\exp\left(-R_c d_{min}\bar{\gamma}_b f_{max}(\bar{\gamma}_b)\right) \\ &= 2^k\exp\left(-d_{min}R_c\bar{\gamma}_b f_{max}(\bar{\gamma}_b)\right) \end{aligned}$$

which shows the equivalence between the upper bounds given in (b) and (c).
In order for the bound to go to zero, as k is increased to infinity we need the rest of the argument of the exponent to be negative, or

$$(\beta d_{min}\bar{\gamma}_b/ne - \ln 2) > 0 \Rightarrow \bar{\gamma}_b > \frac{ne}{d_{min}\beta}\ln 2 \Rightarrow \bar{\gamma}_{b min} = \frac{2e}{\beta}\ln 2$$

Replacing for the values of β found in part (a) we get:

$$\begin{aligned}
\bar{\gamma}_{b min,PSK} &= -0.96 \text{ dB} \\
\bar{\gamma}_{b min,DPSK} &= 2.75 \text{ dB} \\
\bar{\gamma}_{b min,non-coh.FSK} &= 5.76 \text{ dB}
\end{aligned}$$

As expected, among the three, binary PSK has the least stringent SNR requirement for asymptotic performance.

CHAPTER 15

Problem 15.1 :

$$g_k(t) = e^{j\theta_k} \sum_{n=0}^{L-1} a_k(n) p(t - nT_c)$$

The unit energy constraint is :

$$\int_0^T g_k(t) g_k^*(t) dt = 1$$

We also define as cross-correlation :

$$\rho_{ij}(\tau) = \int_0^T g_i(t) g_j^*(t - \tau) dt$$

(a) For synchronous transmission, the received lowpass-equivalent signal $r(t)$ is again given by (15-3-9), while the log-likelihood ratio is :

$$\begin{aligned} \Lambda(\mathbf{b}) &= \int_0^T \left| r(t) - \sum_{k=1}^K \sqrt{E_k} b_k g_k(t) \right|^2 dt \\ &= \int_0^T |r(t)|^2 dt + \sum_k \sum_j \sqrt{E_k} \sqrt{E_j} b_j b_k^* \int_0^T g_j(t) g_k^*(t) dt \\ &\quad -2Re \left[\sum_{k=1}^K \sqrt{E_k} b_k \int_0^T r(t) g_k^*(t) \right] \\ &= \int_0^T |r(t)|^2 dt + \sum_k \sum_j \sqrt{E_k} \sqrt{E_j} b_j b_k \rho_{jk}(0) \\ &\quad -2Re \left[\sum_{k=1}^K \sqrt{E_k} b_k r_k \right] \end{aligned}$$

where $r_k = \int_0^T r(t) g_k^*(t) dt$, and we assume that the information sequence $\{b_k\}$ is real. Hence, the correlation metrics can be expressed in a similar form to (15-3-15) :

$$C(\mathbf{r}_k, \mathbf{b}_k) = 2\mathbf{b}_K^t Re(\mathbf{r}_K) - \mathbf{b}_K^t \mathbf{R}_s \mathbf{b}_K$$

The only difference from the real-valued case of the text is that the correlation matrix R_s uses the complex-valued cross-correlations given above :

$$\mathbf{R}_s[ij] = \left\{ \begin{array}{ll} \rho_{ij}^*(0), & i \leq j \\ \rho_{ij}(0), & i > j \end{array} \right\}$$

and that the matched filters producing $\{r_k\}$ employ the complex-conjugate of the signature waveforms $\{g_k(t)\}$.

(b) Following the same procedure as in pages 852-853 of the text, we see that the correlator outputs are :

$$r_k(i) = \int_{iT+\tau_k}^{(i+1)T+\tau_k} r(t)g_k^*(t - iT - \tau_k)dt$$

and that these can be expressed in matrix form as :

$$\mathbf{r} = \mathbf{R}_N \mathbf{b} + \mathbf{n}$$

where **r**,**b**,**n** are given by (15-3-20)-(15-3-22) and :

$$\mathbf{R}_N = \begin{bmatrix} \mathbf{R_a}(0) & \mathbf{R_a}(-1) & \mathbf{0} & \cdots & \cdots \\ \mathbf{R_a}(1) & \mathbf{R_a}(0) & \mathbf{R_a}(-1) & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R_a}(1) & \mathbf{R_a}(0) \end{bmatrix} \Rightarrow$$

$$\mathbf{R}_N = \begin{bmatrix} \mathbf{R_a}(0) & \mathbf{R_a}^H(1) & \mathbf{0} & \cdots & \cdots \\ \mathbf{R_a}(1) & \mathbf{R_a}(0) & \mathbf{R_a}^H(1) & \mathbf{0} & \cdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{R_a}(1) & \mathbf{R_a}(0) \end{bmatrix}$$

where $\mathbf{R}_a(m)$ is a $K \times K$ matrix with elements :

$$R_{kl}(m) = \int_{-\infty}^{\infty} g_k^*(t - \tau_k)g_l(t + mT - \tau_l)dt$$

and we have exploited the fact (which holds in the real-valued case, too) that :

$$\mathbf{R_a}(m) = \mathbf{R_a^{*t}}(-m) = \mathbf{R_a^H}(-m)$$

Finally, we note that $\mathbf{R}_a(0) = \mathbf{R}_s$, the correlation matrix of the real-valued case.

Problem 15.2 :

The capacity per user C_K is :

$$C_K = \frac{1}{K} W \log_2 \left(1 + \frac{P}{WN_0}\right), \quad \lim_{K \to \infty} C_K = 0$$

and the total capacity :

$$KC_K = W \log_2 \left(1 + \frac{P}{WN_0}\right)$$

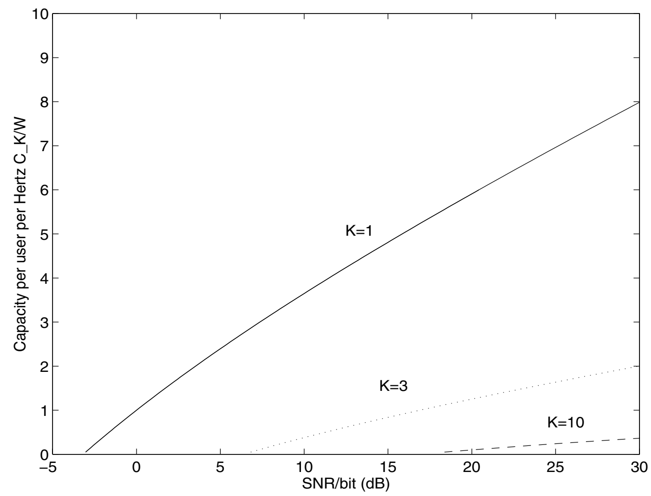
which is independent of K. By using the fact that : $P = C_K \mathcal{E}_b$ we can rewrite the above equations as :

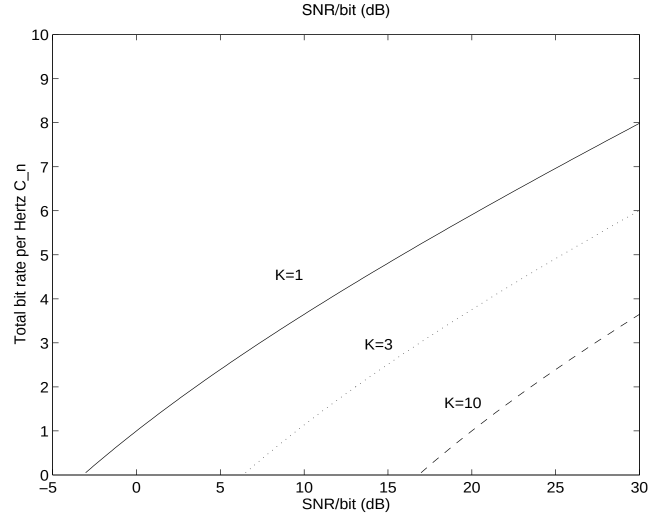
$$C_K = \frac{1}{K} W \log_2 \left(1 + \frac{C_K \mathcal{E}_b}{W N_0}\right) \Rightarrow$$
$$K \frac{C_K}{W} = \log_2 \left(1 + \frac{C_K \mathcal{E}_b}{W N_0}\right) \Rightarrow$$
$$\frac{\mathcal{E}_b}{N_0} = \frac{\left(2^K\right)^{\frac{C_K}{W}} - 1}{\frac{C_K}{W}}$$

which is the relationship between the SNR and the normalized capacity per user. The relationship between the normalized total capacity $C_n = K \frac{C_K}{W}$ and the SNR is :

$$\frac{\mathcal{E}_b}{N_0} = K \frac{2^{C_n} - 1}{C_n}$$

The corresponding plots for these last two relationships are given in the following figures :





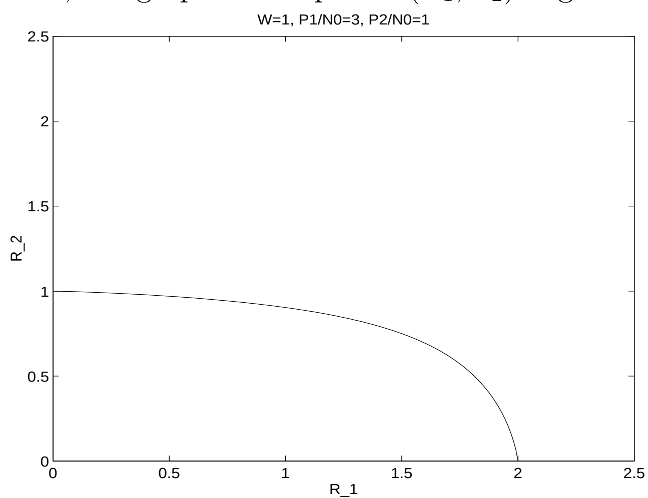
As we observe the normalized capacity per user C_K/W decreases to 0 as the number of user increases. On the other hand, we saw that the total normalized capacity C_n is constant, independent of the number of users K. The second graph is explained by the fact that as the number of users increases, the capacity per user C_K, decreases and hence, the SNR/bit=P/C_K increases, for the same user power P. That's why the curves are shifted to the right, as $K \to \infty$.

Problem 15.3 :

(a)

$$C_1 = aW \log_2 \left(1 + \frac{P_1}{aWN_0}\right)$$
$$C_2 = (1-a)W \log_2 \left(1 + \frac{P_2}{(1-a)WN_0}\right)$$
$$C = C_1 + C_2 = W \left[a \log_2 \left(1 + \frac{P_1}{aWN_0}\right) + (1-a) \log_2 \left(1 + \frac{P_2}{(1-a)WN_0}\right)\right]$$

As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure:



(b) Substituting $P_1/a = P_2/(1-a) = P_1 + P_2$, in the expression for $C = C_1 + C_2$, we obtain :

$$\begin{aligned} C = C_1 + C_2 &= W \left[a \log_2 \left(1 + \tfrac{P_1+P_2}{WN_0}\right) + (1-a) \log_2 \left(1 + \tfrac{P_1+P_2}{WN_0}\right)\right] \\ &= W \log_2 \left(1 + \tfrac{P_1+P_2}{WN_0}\right) \end{aligned}$$

which is the maximum rate that can be satisfied, based on the inequalities that the rates R_1, R_2 must satisfy. Hence, the distribution of the bandwidth according to the SNR of each user, produces the maximum achievable rate.

Problem 15.4 :

(a) Since the transmitters are peak-power-limited, the constraint on the available power holds

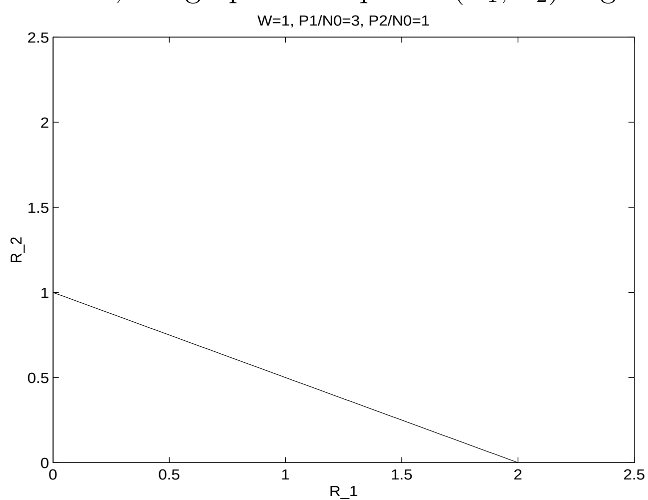
for the allocated time frame when each user transmits. This is more restrictive that an average-power limited TDMA system, where the power is averaged over all the time frames, so each user can transmit in his allocated frame with power P_i/a_i, where a_i is the fraction of the time that the user transmits.
Hence, in the peak-power limited system :

$$C_1 = aW\log_2\left(1+\frac{P_1}{WN_0}\right)$$

$$C_2 = (1-a)W\log_2\left(1+\frac{P_2}{WN_0}\right)$$

$$C = C_1 + C_2 = W\left[a\log_2\left(1+\frac{P_1}{WN_0}\right) + (1-a)\log_2\left(1+\frac{P_2}{WN_0}\right)\right]$$

(b) As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure



We note that the peak-power-limited TDMA system has a more restricted achievable region (R_1, R_2). compared to the FDMA system of problem 15.3.

Problem 15.5 :

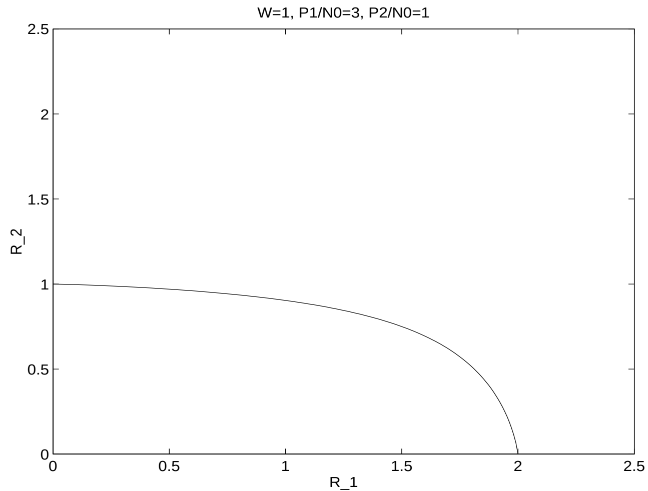
(a) Since the system is average-power limited, the i-th user can transmit in his allocated time-frame with peak-power P_i/a_i, where a_i is the fraction of the time that the user transmits.
Hence, in the average-power limited system :

$$C_1 = aW\log_2\left(1+\frac{P_1/a}{WN_0}\right)$$

$$C_2 = (1-a)W\log_2\left(1+\frac{P_2/(1-a)}{WN_0}\right)$$

$$C = C_1 + C_2 = W\left[a\log_2\left(1+\frac{P_1}{aWN_0}\right) + (1-a)\log_2\left(1+\frac{P_2}{(1-a)WN_0}\right)\right]$$

(b) As a varies between 0 and 1, the graph of the points (C_1, C_2) is given in the following figure



(c) We note that the expression for the total capacity is the same as that of the FDMA in Problem 15.2. Hence, if the time that each user transmits is proportional to the transmitter's power : $P_1/a = P_2/(1-a) = P_1 + P_2$, we have :

$$\begin{aligned} C = C_1 + C_2 &= W\left[a\log_2\left(1+\tfrac{P_1+P_2}{WN_0}\right) + (1-a)\log_2\left(1+\tfrac{P_1+P_2}{WN_0}\right)\right] \\ &= W\log_2\left(1+\tfrac{P_1+P_2}{WN_0}\right) \end{aligned}$$

which is the maximum rate that can be satisfied, based on the inequalities that the rates R_1, R_2 must satisfy. Hence, the distribution of the time that each user transmits according to the respective SNR produces the maximum achievable rate.

Problem 15.6 :

(a) We have

$$r_1 = \int_0^T r(t) g_1(t) dt$$

Since $\int_0^T g_1(t)g_1(t) = 1$, and $\int_0^T g_1(t)g_2(t) = \rho$

$$r_1 = \sqrt{\mathcal{E}_1} b_1 + \sqrt{\mathcal{E}_2} b_2 \rho + n_1$$

where $n_1 = \int_0^T n(t)g_1(t)dt$. Similarly

$$r_2 = \sqrt{\mathcal{E}_1} b_1 \rho + \sqrt{\mathcal{E}_2} b_2 + n_2$$

where $n_2 = \int_0^T n(t)g_2(t)dt$

(b) We have $E[n_1]$ $(= m_1) = E[n_2]$ $(= m_2) = 0$. Hence

$$\begin{aligned} \sigma_1^2 = E[n_1^2] &= E\left[\int_0^T \int_0^T g_1(a)g_1(b)n(a)n(b)dadb\right] \\ &= \frac{N_0}{2} \int_0^T g_1(a)g_1(a)da \\ &= \frac{N_0}{2} \end{aligned}$$

In the same way, $\sigma_1^2 = E[n_1^2] = \frac{N_0}{2}$. The covariance is equal to

$$\begin{aligned} \mu_{12} = E[n_1 n_2] - E[n_1]E[n_2] &= E[n_1 n_2] \\ &= E\left[\int_0^T \int_0^T g_1(a)g_2(b)n(a)n(b)dadb\right] \\ &= \frac{N_0}{2} \int_0^T g_1(a)g_2(a)da \\ &= \frac{N_0}{2}\rho \end{aligned}$$

(c) Given b_1 and b_2, then (r_1,r_2) follow the pdf of (n_1,n_2) which are jointly Gaussian with a pdf given by (2-1-150) or (2-1-156). Using the results from (b)

$$\begin{aligned} p(r_1, r_2|b_1, b_2) &= p(n_1, n_2) \\ &= \frac{1}{2\pi \frac{N_0}{2}\sqrt{1-\rho^2}} \exp\left[-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right] \end{aligned}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1} b_1 - \sqrt{\mathcal{E}_2} b_2 \rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2} b_2 - \sqrt{\mathcal{E}_1} b_1 \rho$

Problem 15.7 :

We use the result for r_1, r_2 from Problem 5.6 (a) (or the equivalent expression (15.3-40)). Then, assuming $b_1 = 1$ was transmitted, the probability of error for b_1 is

$$\begin{aligned} P_1 &= P(error_1|b_2 = 1)P(b_2 = 1) + P(error_1|b_2 = -1)P(b_2 = -1) \\ &= Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_1} + \rho\sqrt{\mathcal{E}_2})^2}{N_0}}\right)\frac{1}{2} + Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_1} - \rho\sqrt{\mathcal{E}_2})^2}{N_0}}\right)\frac{1}{2} \end{aligned}$$

The same expression is obtained when $b_1 = -1$ is transmitted. Hence

$$P_1 = \frac{1}{2}Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_1} + \rho\sqrt{\mathcal{E}_2})^2}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_1} - \rho\sqrt{\mathcal{E}_2})^2}{N_0}}\right)$$

Similarly

$$P_2 = \frac{1}{2}Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_2}+\rho\sqrt{\mathcal{E}_1})^2}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{(\sqrt{\mathcal{E}_2}-\rho\sqrt{\mathcal{E}_1})^2}{N_0}}\right)$$

Problem 15.8 :

(a)

$$P(b_1,b_2|r(t),0\le t\le T) = \frac{p(r(t),0\le t\le T|b_1,b_2)P(b_1,b_2)}{p(r(t),0\le t\le T)}$$

But $P(b_1,b_2)=P(b_1)P(b_2)=1/4$ for any pair of (b_1,b_2) and $p(r(t),0\le t\le T)$ is independent of (b_1,b_2). Hence

$$\arg\max_{b_1,b_2} P(b_1,b_2|r(t),0\le t\le T) = \arg\max_{b_1,b_2} p(r(t),0\le t\le T|b_1,b_2)$$

which shows the equivalence between the MAP and ML criteria, when b_1, b_2 are equiprobable.

(b) Sufficient statistics for $r(t), 0\le t\le T$ are the correlator outputs r_1, r_2 at $t=T$. From Problem 15.6 the joint pdf of r_1, r_2 given b_1, b_2 is

$$p(r_1,r_2|b_1,b_2) = \frac{1}{2\pi\frac{N_0}{2}\sqrt{1-\rho^2}}\exp\left\{-\frac{x_1^2-2\rho x_1x_2+x_2^2}{2(1-\rho^2)}\right\}$$

where $x_1 = r_1-\sqrt{\mathcal{E}_1}b_1-\sqrt{\mathcal{E}_2}b_2\rho$ and $x_2 = r_2-\sqrt{\mathcal{E}_2}b_2-\sqrt{\mathcal{E}_1}b_1\rho$

The ML detector searches for the arguments b_1, b_2 that maximize $p(r_1,r_2|b_1,b_2)$. We see that the term outside the exponent and the demoninator of the exponent do not depend on b_1, b_2. Hence :

$$\begin{aligned}(b_1,b_2) &= \arg\max\exp\left[-(x_1^2-2\rho x_1x_2+x_2^2)\right]\\ &= \arg\max\left[-(x_1^2-2\rho x_1x_2+x_2^2)\right]\end{aligned}$$

Expanding x_1, x_2 and remembering that additive terms which are constant independent of b_1, b_2 (e.g. r_i^2, or $b_i^2\ (=1)$) do not affect the argument of the maximum, we arrive at

$$\begin{aligned}(b_1,b_2) &= \arg\max\left(2(1-\rho^2)\sqrt{\mathcal{E}_1}b_1r_1+2(1-\rho^2)\sqrt{\mathcal{E}_2}b_2r_2-2(1-\rho^2)\sqrt{\mathcal{E}_1\mathcal{E}_2}b_1b_2\rho\right)\\ &= \arg\max\left(\sqrt{\mathcal{E}_1}b_1r_1+\sqrt{\mathcal{E}_2}b_2r_2-\sqrt{\mathcal{E}_1\mathcal{E}_2}b_1b_2\rho\right)\end{aligned}$$

Problem 15.9 :

(a)

$$\begin{aligned}P(b_1|r(t),0\le t\le T) &= P(b_1|r_1,r_2)\\ &= P(b_1,b_2=1|r_1,r_2)+P(b_1,b_2=-1|r_1,r_2)\end{aligned}$$

But

$$P(b_1, b_2 = x|r_1, r_2) = \frac{p(r_1, r_2|b_1, b_2 = x)}{p(r_1, r_2)} P(b_1, b_2 = x)$$

and $p(r_1, r_2)$ and $P(b_1, b_2 = x)$ do not depend on the value of b_1. Hence

$$\arg\max_{b_1} P(b_1|r(t), 0 \le t \le T) = \arg\max_{b_1} (p(r_1, r_2|b_1, b_2 = 1) + p(r_1, r_2|b_1, b_2 = -1))$$

From Problem 15.6 the joint pdf of r_1, r_2 given b_1, b_2 is

$$p(r_1, r_2|b_1, b_2) = \frac{1}{2\pi\frac{N_0}{2}\sqrt{1-\rho^2}} \exp\left\{-\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)}\right\}$$

where $x_1 = r_1 - \sqrt{\mathcal{E}_1}b_1 - \sqrt{\mathcal{E}_2}b_2\rho$ and $x_2 = r_2 - \sqrt{\mathcal{E}_2}b_2 - \sqrt{\mathcal{E}_1}b_1\rho$. Expanding x_1, x_2 and remembering that additive terms which are constant independent of b_1, b_2 (e.g. r_i^2, or b_i^2 $(=1)$) do not affect the argument of the maximum, we arrive at

$$\begin{aligned}
\arg\max_{b_1} P(b_1|r(t), 0 \le t \le T) &= \arg\max\Big[\exp\Big(\tfrac{\sqrt{\mathcal{E}_1}b_1r_1 + \sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0}\Big) \\
&\quad + \exp\Big(\tfrac{\sqrt{\mathcal{E}_1}b_1r_1 - \sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0}\Big)\Big] \\
&= \arg\max\Big[\exp(\tfrac{\sqrt{\mathcal{E}_1}b_1r_1}{N_0}) \\
&\quad \times \Big(\exp(\tfrac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0}) + \exp(\tfrac{-\sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0})\Big)\Big] \\
&= \arg\max\Big[\exp(\tfrac{\sqrt{\mathcal{E}_1}b_1r_1}{N_0}) \cdot 2\cosh(\tfrac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0})\Big] \\
&= \arg\max\Big[\tfrac{\sqrt{\mathcal{E}_1}b_1r_1}{N_0} + \ln\cosh(\tfrac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1\rho}{N_0})\Big]
\end{aligned}$$

(b) From part(a)

$$\begin{aligned}
b_1 = 1 \quad &\Leftrightarrow \quad \tfrac{\sqrt{\mathcal{E}_1}r_1}{N_0} + \ln\cosh(\tfrac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}) > \\
&\qquad \tfrac{-\sqrt{\mathcal{E}_1}r_1}{N_0} + \ln\cosh(\tfrac{\sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}) \\
&\Leftrightarrow \quad 2\tfrac{\sqrt{\mathcal{E}_1}r_1}{N_0} + \ln\left(\frac{\cosh(\frac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0})}{\cosh(\frac{\sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0})}\right) > 0
\end{aligned}$$

Hence

$$b_1 = sgn\left[r_1 - \frac{N_0}{2\sqrt{\mathcal{E}_1}}\ln\left(\frac{\cosh(\frac{\sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0})}{\cosh(\frac{\sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0})}\right)\right]$$

Problem 15.10 :

As $N_0 \to 0$, the probability in expression (15.3-62) will be dominated by the term which has the smallest argument in the Q function. Hence

$$effective\ SNR = \min_{b_j} \frac{\left[\sqrt{\mathcal{E}_k} + \sum_{j\neq k}\sqrt{\mathcal{E}_j}b_j\rho_{jk}\right]^2}{N_0}$$

The minimum over b_j is achieved when all terms add destructively to the $\sqrt{\mathcal{E}_k}$ term (or, it is 0, if the term inside the square is negative). Therefore

$$\eta_k = \left[\max\left\{0, 1 - \sum_{j\neq k}\sqrt{\frac{\mathcal{E}_j}{\mathcal{E}_k}}|\rho_{jk}|\right\}\right]^2$$

Problem 15.11 :

The probability that the ML detector makes an error for the first user is :

$$\begin{array}{rcl} P_1 & = & \sum_{b_1,b_2} P(\hat{b}_1 \neq b_1|b_1,b_2)(P(b_1,b_2) \\ & = & \frac{1}{4}(P[++ \to -+] + P[++ \to --]) \\ & + & \frac{1}{4}(P[-+ \to ++] + P[-+ \to +-]) \\ & + & \frac{1}{4}(P[+- \to --] + P[-+ \to -+]) \\ & + & \frac{1}{4}(P[-- \to +-] + P[-+ \to ++]) \end{array}$$

where $P[b_1 b_2 \to \hat{b}_1\hat{b}_2]$ denotes the probability that the detector chooses $(\hat{b}_1\hat{b}_2)$ conditioned on (b_1, b_2) having being transmitted. Due to the symmetry of the decision statistic, the above relationship simplifies to

$$\begin{array}{rcl} P_1 & = & \frac{1}{2}(P[-- \to +-] + P[-- \to ++]) \\ & + & \frac{1}{2}(P[-+ \to ++] + P[-+ \to +-]) \end{array} \quad (1)$$

From Problem 15.8 we know that the decision of this detector is based on

$$(\hat{b}_1, \hat{b}_2) = \arg\max\left(S(b_1,b_2) = \sqrt{\mathcal{E}_1}b_1 r_1 + \sqrt{\mathcal{E}_2}b_2 r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}b_1 b_2\rho\right)$$

Hence, $P[-- \to +-]$ can be upper bounded as

$$P[-- \to +-] \leq P[S(--) < S(+-)|(--) \text{ transmitted}]$$

This is a bound and not an equality since the if $S(--) < S(+-)$ then $(--)$ is not chosen, but not necessarily in favor of $(+-)$; it may have been in favor of $(++)$ or $(-+)$.
The last bound is easy to calculate :

$$P[S(--) < S(+-)|(--)\text{transmitted}]$$

$$\begin{array}{rcl} & = P & [-\sqrt{\mathcal{E}_1}r_1 - \sqrt{\mathcal{E}_2}r_2 - \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho < \sqrt{\mathcal{E}_1}r_1 - \sqrt{\mathcal{E}_2}r_2 + \sqrt{\mathcal{E}_1\mathcal{E}_2}\rho \\ & & |r_1 = -\sqrt{\mathcal{E}_1} - \sqrt{\mathcal{E}_2}\rho + n_1; r_1 = -\sqrt{\mathcal{E}_1} - \sqrt{\mathcal{E}_2}\rho + n_1] \\ & = P & [n_1 > \sqrt{\mathcal{E}_1}] = Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) \end{array}$$

Similarly, for the other three terms of (1) we obtain :

$$\begin{array}{rcl} P[-- \rightarrow ++] & \leq & P[S(--) < S(++)|(--) \text{ transmitted}] \\ & = & P[\sqrt{\mathcal{E}_1}n_1 + \sqrt{\mathcal{E}_2}n_2 > \mathcal{E}_1 + \mathcal{E}_2 + 2\sqrt{\mathcal{E}_1\mathcal{E}_1}\rho] \\ & = & Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2+2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right) \end{array}$$

$$\begin{array}{rcl} P[-+ \rightarrow +-] & \leq & P[S(-+) < S(+-)|(-+) \text{ transmitted}] \\ & = & P[\sqrt{\mathcal{E}_1}n_1 - \sqrt{\mathcal{E}_2}n_2 > \mathcal{E}_1 + \mathcal{E}_2 - 2\sqrt{\mathcal{E}_1\mathcal{E}_1}\rho] \\ & = & Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2-2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right) \end{array}$$

$$\begin{array}{rcl} P[-+ \rightarrow ++] & \leq & P[S(-+) < S(++)|(-+) \text{ transmitted}] \\ & = & P[n_1 > \sqrt{\mathcal{E}_1}] \\ & = & Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) \end{array}$$

By adding the four terms we obtain

$$\begin{array}{rcl} P_1 & \leq & Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2-2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right) \\ & & +\frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2+2\sqrt{\mathcal{E}_1\mathcal{E}_2}\rho}{N_0}}\right) \end{array}$$

But we note that if $\rho \geq 0$, the last term is negligible, while if $\rho \leq 0$, then the second term is negiligible. Hence, the bound can be written as

$$P_1 \leq Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2-2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0}}\right)$$

Problem 15.12 :

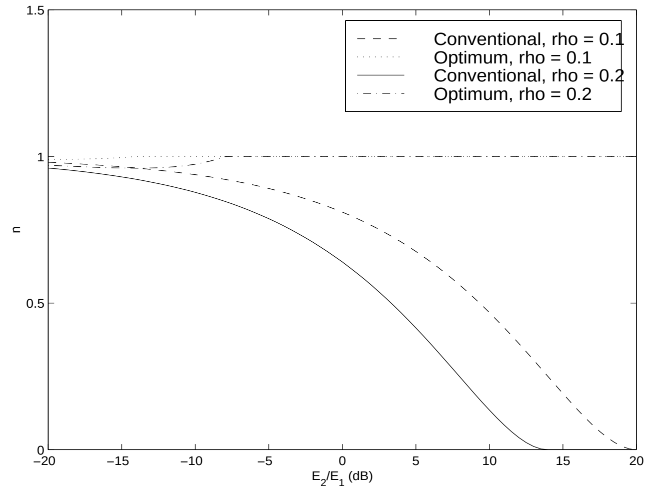
(a) We have seen in Prob. 15.11 that the probability of error for user 1 can be upper bounded by

$$P_1 \leq Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}}\right) + \frac{1}{2}Q\left(\sqrt{2\frac{\mathcal{E}_1+\mathcal{E}_2-2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0}}\right)$$

As $N_0 \rightarrow 0$ the probability of error will be dominated by the Q function with the smallest argument. Hence

$$\begin{array}{rcl} \eta_1 & = & \min\left\{(\frac{2\mathcal{E}_1}{N_0})/(\frac{2\mathcal{E}_1}{N_0}), 2\frac{\mathcal{E}_1+\mathcal{E}_2-2\sqrt{\mathcal{E}_1\mathcal{E}_2}|\rho|}{N_0}/(\frac{2\mathcal{E}_1}{N_0})\right\} \\ & = & \min\left\{1, 1+\frac{\mathcal{E}_2}{\mathcal{E}_1} - 2\sqrt{\frac{\mathcal{E}_2}{\mathcal{E}_1}}|\rho|\right\} \end{array}$$

(b) The plot of the asymptotic efficiencies is given in the following figure



We notice the much better performance of the optimal detector especially when the interfearer (user 2) is much stronger than the signal. We also notice that the performance of the conventional detector decreases as $|\rho|$ (i.e interference) increases, which agrees with the first observation.

Problem 15.13 :

The decision rule for the decorrelating detector is $\hat{\mathbf{b}}_\mathbf{2} = sgn(\mathbf{b}_\mathbf{2}^\mathbf{0})$, where $\mathbf{b}_\mathbf{2}^\mathbf{0}$ is the output of the decorrelating operation as given by equation (15.3-41). The signal component for the first term in the equation is $\sqrt{\mathcal{E}_1}$. The noise component is

$$n = \frac{n_1 - \rho n_2}{1 - \rho^2}$$

with variance

$$\begin{aligned} \sigma_1^2 = E[n^2] &= \frac{E[n_1 - \rho n_2]^2}{(1-\rho^2)^2} \\ &= \frac{E[n_1^2] + \rho^2 E[n_2^2] - 2\rho E[n_1 n_2]}{(1-\rho^2)^2} \\ &= \frac{N_0}{2}\frac{1+\rho^2}{(1-\rho^2)^2} \\ &= \frac{N_0}{2}\frac{1}{(1-\rho^2)} \end{aligned}$$

Hence

$$P_1 = Q\left(\sqrt{\frac{2\mathcal{E}_1}{N_0}(1-\rho^2)}\right)$$

Similarly, for the second user

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_2}{N_0}(1-\rho^2)}\right)$$

Problem 15.14 :

(a) The matrix $\mathbf{R_s}$ is

$$\mathbf{R_s} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Hence the linear transformation $\mathbf{A_0}$ for the two users will be

$$\mathbf{A_0} = \left(\mathbf{R_s} + \frac{N_0}{2}\mathbf{I}\right)^{-1} = \begin{bmatrix} 1+\frac{N_0}{2} & \rho \\ \rho & 1+\frac{N_0}{2} \end{bmatrix}^{-1} = \frac{1}{\left(1+\frac{N_0}{2}\right)^2 - \rho^2} \begin{bmatrix} 1+\frac{N_0}{2} & -\rho \\ -\rho & 1+\frac{N_0}{2} \end{bmatrix}$$

(b) The limiting form of A_0, as $N_0 \to 0$ is obviously

$$A_0 \to \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

which is the same as the transformation for the decorrelating detector, as given by expression (15.3-37).

(c) The limiting form of A_0, as $N_0 \to \infty$ is

$$A_0 \approx \frac{1}{\left(\frac{N_0}{2}\right)^2} \begin{bmatrix} \frac{N_0}{2} & -\rho \\ -\rho & 1+\frac{N_0}{2} \end{bmatrix} \to \frac{1}{\left(\frac{N_0}{2}\right)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is simply a (scaled) form of the conventional single-user detector, since the decision for each user is based solely on the output of the particular user's correlator.

Problem 15.15 :

(a) The performance of the receivers, when no post-processing is used, is the performance of the conventional multiuser detection.

(b) Since : $y_1(l) = b_1(l)w_1 + b_2(l)\rho_{12}^{(1)} + b_2(l-1)\rho_{21}^{(1)} + n$, the decision variable $z_1(l)$, for the first user after post-processing, is equal to :

$$z_1(l) = b_1(l)w_1 + n + \rho_{21}^{(1)} e_2(l-1) + \rho_{12}^{(1)} e_2(l)$$

where n is Gaussian with zero mean and variance $\sigma^2 w_1$ and, by definition : $e_2(l) \equiv b_2(l) - sgn\left[y_2(l)\right]$. We note that $e_2(l)$ is not orthogonal to $e_2(l-1)$, in general; however these two quantities are orthogonal when conditioned on $b_1(l)$. The distribution of $e_2(l-1)$, conditioned on $b_1(l)$ is :

$$\begin{aligned}
P\left[e_2(l-1)=+2|b_1(l)\right] &= \tfrac{1}{4}Q\left[\tfrac{w_2+\rho_{12}^{(2)}+\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]+\tfrac{1}{4}Q\left[\tfrac{w_2-\rho_{12}^{(2)}+\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right] \\
P\left[e_2(l-1)=-2|b_1(l)\right] &= \tfrac{1}{4}Q\left[\tfrac{w_2-\rho_{12}^{(2)}-\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]+\tfrac{1}{4}Q\left[\tfrac{w_2+\rho_{12}^{(2)}-\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right] \\
P\left[e_2(l-1)=0|b_1(l)\right] &= 1-P\left[e_2(l-1)=2|b_1(l)\right]-P\left[e_2(l-1)=-2|b_1(l)\right]
\end{aligned}$$

The distribution of $e_2(l)$, given $b_1(l)$, is similar, just exchange $\rho_{12}^{(2)}$ with $\rho_{21}^{(2)}$. Then, the probability of error for user 1 is :

$$P\left[\hat{b}_1(l)\neq b_1(l)\right]=\sum_{\substack{a\in\{-2,0,2\}\\ b\in\{-1,+1\}\\ c\in\{-2,0,2\}}}\tfrac{1}{2}P\left[e_2(l-1)=a|b_1(l)=b\right]P\left[e_2(l)=c|b_1(l)=b\right]\times$$
$$\times Q\left[\frac{w_1+\left(\rho_{12}^{(1)}c+\rho_{21}^{(1)}a\right)b_1(l)}{\sigma\sqrt{w_1}}\right]$$

The distribution of $e_2(l-1)$, conditioned on $b_1(l)$, when $\sigma\to 0$ is :

$$\begin{aligned}
P\left[e_2(l-1)=a|b_1(l)\right] &\approx \tfrac{1}{4}Q\left[\frac{w_2-\left|\rho_{12}^{(2)}\right|+a\rho_{21}^{(2)}b_1(l)/2}{\sigma\sqrt{w_2}}\right], \quad a=\pm 2 \\
P\left[e_2(l-1)=0|b_1(l)\right] &= 1-P\left[e_2(l-1)=2|b_1(l)\right]-P\left[e_2(l-1)=-2|b_1(l)\right]
\end{aligned}$$

This distribution may be concisely written as :

$$P\left[e_2(l-1)=a|b_1(l)\right]\approx\frac{1}{4}Q\left[\frac{|a|}{2}\frac{w_2-\left|\rho_{12}^{(2)}\right|+\frac{a}{2}\rho_{21}^{(2)}b_1(l)}{\sigma\sqrt{w_2}}\right]$$

which is exponentially tight. The limiting form of the probability of error is (dropping constants)

$$P\left[\hat{b}_1(l)\neq b_1(l)\right]\approx\sum_{\substack{a\in\{-2,0,2\}\\ b\in\{-1,+1\}\\ c\in\{-2,0,2\}}}Q\left[\frac{|a|}{2}\frac{w_2-\left|\rho_{12}^{(2)}\right|+\frac{a}{2}\rho_{21}^{(2)}b}{\sigma\sqrt{w_2}}\right]\times$$
$$\times Q\left[\frac{|c|}{2}\frac{w_2-\left|\rho_{21}^{(2)}\right|+\frac{c}{2}\rho_{12}^{(2)}b}{\sigma\sqrt{w_2}}\right]\times Q\left[\frac{w_1+\left(\rho_{12}^{(1)}c+\rho_{21}^{(1)}a\right)b}{\sigma\sqrt{w_1}}\right]$$

(c) Consider the special case :

$$\begin{aligned}
sgn\left(\rho_{12}^{(1)}\right) &= sgn\left(\rho_{12}^{(2)}\right) \\
sgn\left(\rho_{21}^{(1)}\right) &= sgn\left(\rho_{21}^{(2)}\right)
\end{aligned}$$

as would occur for far-field transmission (this case is the most prevalent in practice ; other cases follow similarly). Then, the slowest decaying term corresponds to either :

$$sgn\left(b\rho_{21}^{(1)}a\right) = sgn\left(b\rho_{12}^{(1)}c\right) = -1$$

for which the resulting term is :

$$Q\left[\sqrt{\frac{w_1}{\sigma^2}}\sqrt{2}\left\{\sqrt{\frac{w_2}{w_1}} - \frac{\left|\rho_{12}^{(2)}\right| + \left|\rho_{21}^{(2)}\right|}{\sqrt{w_1}\sqrt{w_2}}\right\}\right] \cdot Q\left[\sqrt{\frac{w_1}{\sigma^2}}\left\{1 - 2\frac{\left|\rho_{12}^{(1)}\right| + \left|\rho_{21}^{(1)}\right|}{w_1}\right\}\right]$$

or the case when either ma or $c = 0$. In this case the term is :

$$Q\left[\sqrt{\frac{w_1}{\sigma^2}}\left\{\sqrt{\frac{w_2}{w_1}} - \frac{\left|\rho_{12}^{(2)}\right| + \left|\rho_{21}^{(2)}\right|}{\sqrt{w_1}\sqrt{w_2}}\right\}\right] \cdot Q\left[\sqrt{\frac{w_1}{\sigma^2}}\left\{1 - 2\frac{\max\left(\left|\rho_{12}^{(1)}\right|, \left|\rho_{21}^{(1)}\right|\right)}{w_1}\right\}\right]$$

or the case when $a = c = 0$ for which the term is :

$$Q\left[\sqrt{\frac{w_1}{\sigma^2}}\right]$$

Therefore, the asymptotic efficiency of this detector is :

$$\eta_1 = \min\left[\begin{array}{c} 1, \max^2\left\{0, \sqrt{\frac{w_2}{w_1}} - \frac{\left|\rho_{12}^{(2)}\right| + \left|\rho_{21}^{(2)}\right|}{\sqrt{w_1}\sqrt{w_2}}\right\} + \max^2\left\{0, 1 - 2\frac{\max\left(\left|\rho_{12}^{(1)}\right|, \left|\rho_{21}^{(1)}\right|\right)}{w_1}\right\}, \\ 2\max^2\left\{0, \sqrt{\frac{w_2}{w_1}} - \frac{\left|\rho_{12}^{(2)}\right| + \left|\rho_{21}^{(2)}\right|}{\sqrt{w_1}\sqrt{w_2}}\right\} + \max^2\left\{0, 1 - 2\frac{\max\left(\left|\rho_{12}^{(1)}\right|, \left|\rho_{21}^{(1)}\right|\right)}{w_1}\right\} \end{array}\right]$$

Problem 15.16:

(a) The normalized offered traffic per user is : $G_{user} = \lambda \cdot T_p = \left(\frac{1}{60}\text{ pack/ sec}\right) \cdot \left(\frac{100}{2400}\text{ sec}\right) = 1/1440$. The maximum channel throughput $S_{\max}$ occurs when $G_{\max} = 1/2$; hence, the number of users that will produce the maximum throughput for the system is : $G_{\max}/G_{user} = 720$.

(b) For slotted Aloha, the maximum channel throughput occurs when $G_{\max} = 1$; hence, the number of users that will produce the maximum throughput for the system is : $G_{\max}/G_{user} = 1440$.

Problem 15.17 :

A, the average normalized rate for retransmissions, is the total rate of transmissions (G) times the probability that a packet will overlap. This last probability is equal to the probability that another packet will begin from T_p seconds before until T_p seconds after the start time of the original packet. Since the start times are Poisson-distributed, the probability that the two packets will overlap is $1 - \exp(-2\lambda T_p)$. Hence,

$$A = G(1 - e^{-2G}) \Rightarrow G = S + G(1 - e^{-2G}) \Rightarrow S = Ge^{-2G}$$

Problem 15.18 :

(a) Since the number of arrivals in the interval T, follows a Poisson distribution with parameter λT, the average number of arrivals in the interval T, is $E[k] = \lambda T$.

(b) Again, from the well-known properties of the Poisson distribution : $\sigma^2 = (\lambda T)^2$.

(c)

$$P(k \geq 1) = 1 - P(k = 0) = 1 - e^{-\lambda T}$$

(d)

$$P(k = 1) = \lambda T e^{-\lambda T}$$

Problem 15.19 :

(a) Since the average number of arrivals in 1 sec is $E[k] = \lambda T = 10$, the average time between arrivals is 1/10 sec.

(b)

$$P(\text{at least one arrival within 1 sec}) = 1 - e^{-10} \approx 1$$

$$P(\text{at least one arrival within 0.1 sec}) = 1 - e^{-1} = 0.63$$

Problem 15.20 :

(a) The throughput S and the normalized offered traffic G are related as $S = Ge^{-G} = 0.1$. Solving numerically for G, we find $G = 0.112$.

(b) The average number of attempted transmissions to send a packet, is : $G/S = 1.12$.

Problem 15.21 :

(a)

$$\tau_d = (2 \text{ km}) \cdot \left(5 \; \frac{\mu\text{s}}{\text{km}}\right) = 10 \; \mu\text{s}$$

(b)

$$T_p = \frac{1000 \text{ bits}}{10^7 \text{ bits/ sec}} = 10^{-4} \text{ s}$$

(c)

$$a = \frac{\tau_d}{T_p} = \frac{1}{10}$$

Hence, a carrier-sensing protocol yields a satisfactory performance.

(d) For non-persistent CDMA :

$$S = \frac{Ge^{-aG}}{G(1+2a) + e^{-aG}}$$

The maximum bus utilization occurs when :

$$\frac{dS}{dG} = 0$$

Differentiating the above expression with respect to G, we obtain :

$$e^{-aG} - aG^2(1+2a) = 0$$

which, when solved numerically, gives : $G_{\max} = 2.54$. Then , the maximum throughput will be :

$$S_{\max} = \frac{Ge^{-aG}}{G(1+2a) + e^{-aG}} = 0.515$$

and the maximum bit rate :

$$S_{\max} \cdot 10^7 \text{ bits/ sec} = 5.15 \text{ Mbits/ sec}$$